Mass Problems

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Abstract

A *mass problem* is a set of Turing oracles. If P and Q are mass problems, we say that P is *weakly reducible* to Q if every member of Q Turing computes a member of P. Two mass problems are said to be *weakly equivalent* if each is weakly reducible to the other. A *weak degree* is an equivalence class under weak reducibility. The weak degrees are partially ordered in the obvious way, by weak reducibility. This partial ordering is easily seen to be a complete distributive lattice. We focus on the countable sublattice obtained by restricting to mass problems of the form $P =$ the set of all paths through T, where T is an infinite computable subtree of the full binary tree. We present natural examples of such mass problems arising from mathematical logic, Martin-Löf randomness, effective immunity, and the Arslanov Completeness Criterion. We also present artificial examples constructed by means of priority arguments.

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Contents

1 Turing Degrees

Definition 1.1. We use ω to denote the set of natural numbers:

$$\omega = \{0, 1, 2, \ldots\}.$$

We use ω^ω to denote the set of total functions from ω into ω:

$$\omega^\omega = \{f \mid f : \omega \to \omega\}.$$

We use 2^ω to denote the set of total functions from ω into $\{0, 1\}$:

$$2^\omega = \{X \mid X : \omega \to \{0, 1\}\}.$$

The space ω^ω with the obvious product topology is called *Baire space*. Its subspace 2^ω is called *Cantor space*. We use $f, g, h, \ldots$ to denote points of the Baire space, and $X, Y, Z, \ldots$ to denote points of the Cantor space.

Remark 1.2. We sometimes identify subsets of ω with their characteristic functions in 2^ω. The characteristic function of $A \subseteq \omega$ is $\chi_A \in 2^\omega$ given by

$$\chi_A(n) = \begin{cases} 1 \text{ if } n \in A, \\ 0 \text{ if } n \notin A. \end{cases}$$

Definition 1.3. For $e, m, n \in \omega$ and $g \in \omega^\omega$, we write

$$\{e\}^g(m) = n$$

to mean that e is the Gödel number of a Turing machine which, if started with input m (on the input tape) and oracle g (on the auxiliary or oracle tape), eventually halts with output n (on the output tape).

Remark 1.4. The idea of using an arbitrary, possibly non-computable member of ω^ω as an oracle is due to Turing [44]. See also Rogers [30, Section 9.2].

Definition 1.5. According to Definition 1.3, each $e \in \omega$ gives rise to a *partial recursive functional* Φ from $\omega^\omega \times \omega$ to ω, given by $\Phi(g, m) \simeq \{e\}^g(m)$. We may also view Φ as a partial recursive functional from ω^ω to ω^ω, given by $\Phi(g)(m) \simeq \Phi(g, m) \simeq \{e\}^g(m)$. In either case we say that e is an *index* of Φ.

Remark 1.6. Here $\simeq$ denotes *strong equality* for expressions which may be undefined. Thus $E_1 \simeq E_2$ if and only if E_1, E_2 are both undefined, or both defined and equal. We write $E\downarrow$ to mean that E is defined. We write $E\uparrow$ to mean that E is undefined.

Definition 1.7. For $f, g \in \omega^\omega$, we say that g *Turing computes* f, or f is *recursive in* g, or f is *Turing reducible* to g, abbreviated $f \leq_T g$, if $\Phi(g) = f$ for some partial recursive functional Φ. Thus $f \leq_T g$ if and only if there exists $e \in \omega$ such that $\{e\}^g(m) = f(m)$ for all $m \in \omega$. Clearly $\leq_T$ is a reflexive, transitive relation on ω^ω.

We say that f is *Turing equivalent* to g, abbreviated $f \equiv_T g$, if $f \leq_T g$ and $g \leq_T f$. Clearly $\equiv_T$ is an equivalence relation on ω^ω.

Definition 1.8. The *Turing degrees*, or *degrees of unsolvability*, are the equivalence classes of members of ω^ω under the equivalence relation $\equiv_T$. The Turing degree of $f \in \omega^\omega$ is denoted $\deg_T(f)$. We use $\mathcal{D}_T$ to denote the set of Turing degrees. Thus we have

$$\deg_T(f) = \{g \in \omega^\omega \mid f \equiv_T g\}$$

and

$$\mathcal{D}_T \;=\; \omega^\omega/{\equiv_T} \;=\; \{\deg_T(f) \mid f \in \omega^\omega\}.$$

Definition 1.9. In Definition 1.3, if there is no oracle g, we write simply

$$\{e\}(m) = n.$$

Thus $f \in \omega^\omega$ is said to be *Turing computable*, or *recursive*, if there exists $e \in \omega$ such that $\{e\}(m) = f(m)$ for all $m \in \omega$.

Remark 1.10. We partially order $\mathcal{D}_T$ by putting $\deg_T(f) \leq \deg_T(g)$ if and only if $f \leq_T g$. Under this partial ordering, $\mathcal{D}_T$ is an upper semilattice, with least upper bound operation given by $f \oplus g \in \omega^\omega$, where

$$(f \oplus g)(2n) = f(n), \qquad (f \oplus g)(2n+1) = g(n),$$

for all $f, g \in \omega^\omega$. Moreover $\mathcal{D}_T$ has a bottom element

$$\mathbf{0} = \deg_T(\lambda n.0) = \{f \in \omega^\omega \mid f \text{ is recursive}\}.$$

It can be shown that $\mathcal{D}_T$ has no top element and is not a lattice.

We now compare the Baire space, ω^ω, to the Cantor space, 2^ω.

Theorem 1.11. *For each $f \in \omega^\omega$ there exists $X \in 2^\omega$ such that $f \equiv_T X$.*

Proof. Let X be the characteristic function of $\{2^m 3^n \mid f(m) = n\}$. □

Remark 1.12. In view of Theorem 1.11, the Baire space ω^ω and the Cantor space 2^ω are identical as to Turing degrees. Thus we may write

$$\mathcal{D}_T \;=\; 2^\omega/{\equiv_T} \;=\; \{\deg_T(X) \mid X \in 2^\omega\}.$$

Remark 1.13. The idea behind Turing reducibility is that each non-recursive (i.e., non-computable) $X \in 2^\omega$ is regarded as an "unsolvable problem", viz., the problem of computing X. Then $Y \leq_T X$ means that "the problem" Y is "reducible" to "the problem" X in the sense that, if there were an oracle for solving X, then, with the help of this oracle, we could solve Y. Thus $\deg_T(X)$, the Turing degree of X, is a measure of the "unsolvability" of "the problem" X. In particular, the Turing degree $\mathbf{0}$ corresponds to "solvable problems", i.e., recursive members of 2^ω.

Example 1.14. Turing's original example of an unsolvable problem is the Halting Problem, i.e., the (characteristic function of the) set

$$\begin{aligned} H &= \{e \in \omega \mid \{e\}(0)\downarrow\} \\ &= \{\text{Gödel numbers of Turing machines which eventually halt}\}. \end{aligned}$$

The Turing degree of the Halting Problem is denoted $\mathbf{0}'$. Turing's famous theorem on unsolvability of the Halting Problem amounts to saying that H is nonrecursive, i.e., $\mathbf{0}' > \mathbf{0}$.

In addition, it is known that there are infinitely many Turing degrees $\mathbf{a}$ in the interval $\mathbf{0} \leq \mathbf{a} \leq \mathbf{0}'$. Moreover, the Turing degrees in this interval do not form a lattice.

Definition 1.15. A set $A \subseteq \omega$ is said to be *recursively enumerable*, abbreviated *r. e.*, if $A = \{f(m) \mid m \in \omega\}$ for some recursive function $f : \omega \to \omega$. In this case, an *index* of A is just an index of f. A Turing degree is said to be *recursively enumerable* if it is the Turing degree of (the characteristic function of) an r. e. subset of ω. The set of r. e. Turing degrees is denoted $\mathcal{R}_T$.

Definition 1.16. An r. e. set $C \subseteq \omega$ is said to be *Turing complete* if for every r. e. set $A \subseteq \omega$ we have $A \leq_T C$. An r. e. Turing degree is said to be *Turing complete* if it is the Turing degree of a Turing complete r. e. set.

Remark 1.17. It is known that the halting set H and its Turing degree $\mathbf{0}'$ are recursively enumerable and Turing complete. Moreover, if $\mathbf{a}$ and $\mathbf{b}$ are r. e. Turing degrees, then so is their least upper bound, $\sup(\mathbf{a}, \mathbf{b})$. Thus $\mathcal{R}_T$ is a countable upper semilattice with a top element, $\mathbf{0}'$, and a bottom element, $\mathbf{0}$. It is known that $\mathcal{R}_T$ is infinite and properly included in the interval $\mathbf{0} \leq \mathbf{a} \leq \mathbf{0}'$ in $\mathcal{D}_T$, and is not a lattice.

Remark 1.18. More generally, for any Turing degree $\mathbf{a} = \deg_T(X)$, define $\mathbf{a}' = \deg_T(H^X)$ where H^X is (the characteristic function of)

$$H^X = \{e \mid \{e\}^X(0)\downarrow\},$$

the *halting set* relative to X. It is known that the *Turing jump operator* $\mathbf{a} \mapsto \mathbf{a}'$ is well defined, because $X \leq_T Y$ implies $H^X \leq_T H^Y$. It is known that $\mathbf{a}' > \mathbf{a}$, and that $\mathbf{a}'$ is r. e. relative to $\mathbf{a}$ and Turing complete relative to $\mathbf{a}$.

Remark 1.19. The semilattice $\mathcal{D}_T$ of all Turing degrees, and its subsemilattice $\mathcal{R}_T$ consisting of the r. e. Turing degrees, have been studied intensively for many years. See Sacks [31], Rogers [30], Lerman [23], Simpson [36] [37], Soare [42].

2 Weak and Strong Degrees

Definition 2.1. A *mass problem* is a subset of ω^ω. We use, $P, Q, R, \ldots$ to denote subsets of ω^ω.

Definition 2.2. For $P, Q \subseteq \omega^\omega$, we say that P is *weakly reducible* to Q, abbreviated $P \leq_w Q$, if for each $g \in Q$ there exists $f \in P$ such that $f \leq_T g$. (Recall that $\leq_T$ is Turing reducibility.) The notion of weak reducibility was introduced by Muchnik [28] and has sometimes been called *Muchnik reducibility*. Clearly $\leq_w$ is a reflexive, transitive relation on the powerset of ω^ω.

Definition 2.3. We say that $P, Q \subseteq \omega^\omega$ are *weakly equivalent*, or *Muchnik equivalent*, abbreviated $P \equiv_w Q$, if $P \leq_w Q$ and $Q \leq_w P$. Clearly $\equiv_w$ is an equivalence relation on the powerset of ω^ω. The equivalence classes are called *weak degrees*, or *Muchnik degrees*. The weak degree of P is denoted $\deg_w(P)$. We use $\mathcal{D}_w$ to denote the set of weak degrees. Thus we have

$$\deg_w(P) = \{Q \subseteq \omega^\omega \mid P \equiv_w Q\}$$

and

$$\mathcal{D}_w = \{\deg_w(P) \mid P \subseteq \omega^\omega\}.$$

We partially order $\mathcal{D}_w$ by putting $\deg_w(P) \leq \deg_w(Q)$ if and only if $P \leq_w Q$.

Remark 2.4. The idea behind weak reducibility is that an arbitrary subset of ω^ω may be regarded as a "mass problem", i.e., a problem whose solution is not necessarily unique. If $P \subseteq \omega^\omega$ is viewed a mass problem, the "solutions" of P are just the members of P. A mass problem is regarded as "unsolvable" if it has no recursive solution. Compare Remark 1.13. Viewing P and Q as mass problems, $P \leq_w Q$ means that P is "reducible" to Q in the sense that, given any oracle for a solution of Q, we could use this oracle to compute a solution of P. Thus the weak degree of P is a measure of the "difficulty" of P qua mass problem.

Example 2.5. Let T be a consistent, recursively axiomatizable theory. For instance, we may take $T = \mathsf{PA}$ = first order Peano Arithmetic, or $T = \mathsf{ZF}$ = Zermelo/Fraenkel Set Theory. By a *completion* of T we mean a maximal consistent theory extending T with the same vocabulary as T. Note that T is incomplete if and only if T has more than one completion. The Gödel Incompleteness Theorem implies that this is the case for $T = \mathsf{PA}$ or $T = \mathsf{ZF}$.

We identify sentences with their Gödel numbers in ω. We identify theories as sets of (Gödel numbers of) sentences. Define

$$P_T = \{X \in 2^\omega \mid X \text{ is (the characteristic function of) a completion of } T\}.$$

We may then view P_T as a mass problem, viz., the "problem" of finding a completion of T. The mass problem P_T is regarded as "unsolvable" if and only if the theory T has no decidable completion. This is equivalent to saying that T is *essentially undecidable*, i.e., there is no consistent, decidable theory extending T. By a result of Tarski, this is the case for PA, ZF, and many other theories which arise in the foundations of mathematics.

We now introduce strong reducibility, a variant of weak reducibility.

Definition 2.6. For $P, Q \subseteq \omega^\omega$, we say that P is *strongly reducible* to Q, abbreviated $P \leq_s Q$, if there exists a partial recursive functional $\Phi : Q \to P$, i.e., a partial recursive functional Φ from ω^ω to ω^ω such that the domain of definition of Φ includes Q and for all $g \in Q$ we have $\Phi(g) \in P$. This notion was introduced by Medvedev [27] and has sometimes been called *Medvedev reducibility.* See also Rogers [30, Section 13.7]. Clearly $\leq_s$ is a reflexive, transitive relation on the powerset of ω^ω.

Remark 2.7. Thus strong reducibility is a uniform variant of weak reducibility. Note that $P \leq_s Q$ implies $P \leq_w Q$, but the converse often fails. Later we shall see an analogy

weak reducibility / strong reducibility =

Turing reducibility / truth table reducibility.

See Remark 3.9 below.

Definition 2.8. We define strong degrees in terms of $\leq_s$, just as weak degrees were defined in terms of $\leq_w$ in Definition 2.3.

Explicitly, we say that $P, Q \subseteq \omega^\omega$ are *strongly equivalent*, or *Medvedev equivalent*, abbreviated $P \equiv_s Q$, if $P \leq_s Q$ and $Q \leq_s P$. Clearly $\equiv_s$ is an equivalence relation on the powerset of ω^ω. The equivalence classes are called *strong degrees*, or *Medvedev degrees.* The strong degree of P is denoted $\deg_s(P)$. We use $\mathcal{D}_s$ to denote the set of strong degrees. Thus we have

$$\deg_s(P) = \{Q \subseteq \omega^\omega \mid P \equiv_s Q\}$$

and

$$\mathcal{D}_s = \{\deg_s(P) \mid P \subseteq \omega^\omega\}.$$

We partially order $\mathcal{D}_s$ by putting $\deg_s(P) \leq \deg_s(Q)$ if and only if $P \leq_s Q$.

Remark 2.9. We are mainly interested in weak reducibility and weak degrees. We discuss strong reducibility and strong degrees for technical reasons only.

Theorem 2.10. *The lattices of weak degrees and strong degrees, $\mathcal{D}_w$ and $\mathcal{D}_s$, are distributive lattices with a bottom element.*

Proof. The least upper bound operation for weak or strong reducibility is given by

$$P \times Q = \{f \oplus g \mid f \in P, g \in Q\}.$$

The greatest lower bound operation for weak reducibility is given by $P \cup Q$. The greatest lower bound operation for weak or strong reducibility is given by

$$P + Q = \{\langle 0\rangle^\frown f \mid f \in P\} \cup \{\langle 1\rangle^\frown g \mid g \in Q\}.$$

It is straightforward but instructive to check that the weak degrees form a distributive lattice under these operations. Similarly for the strong degrees.

The bottom element of $\mathcal{D}_w$ is $\mathbf{0} = \deg_w(\omega^\omega)$, and similarly for $\mathcal{D}_s$. Actually, the $\mathbf{0}$ of $\mathcal{D}_w$ and the $\mathbf{0}$ of $\mathcal{D}_s$ are identical, namely

$$\mathbf{0} = \{P \subseteq \omega^\omega \mid P \text{ contains a recursive member}\}.$$

□

Theorem 2.11. *In both $\mathcal{D}_w$ and $\mathcal{D}_s$, the bottom element $\mathbf{0}$ is meet irreducible, i.e., it is not the greatest lower bound of two nonzero degrees.*

Proof. This is obvious, because $P + Q$ (or $P \cup Q$) contains a recursive member if and only if at least one of P and Q contains a recursive member. □

Theorem 2.12. *$\mathcal{D}_T$ is canonically embeddable into $\mathcal{D}_w$ and into $\mathcal{D}_s$. The embeddings preserve order and least upper bound, and carry $\mathbf{0}$ to $\mathbf{0}$.*

Proof. The embedding of $\mathcal{D}_T$ into $\mathcal{D}_w$ is given by $\deg_T(X) \mapsto \deg_w(\{X\})$, and similarly for $\mathcal{D}_s$. Here $\{X\}$ denotes the singleton set whose unique element is X. We have $X \leq_T Y$ if and only if $\{X\} \leq_w \{Y\}$, if and only if $\{X\} \leq_s \{Y\}$. Moreover, $\{X\} \times \{Y\} = \{X \oplus Y\}$. □

Theorem 2.13. *$\mathcal{D}_w$ is a complete distributive lattice.*

Proof. It is straightforward to check that $\mathcal{D}_w$ under $\leq$ is canonically isomorphic to the partial ordering of upward closed subsets of $\mathcal{D}_T$ under reverse inclusion. This is clearly is a complete lattice ordering. The isomorphism is given by $\deg_w(P) \mapsto \{\deg_T(g) \mid P \leq_w \{g\}\}$. □

We now compare the Baire space, ω^ω, to the Cantor space, 2^ω.

Definition 2.14. $P, Q \subseteq \omega^\omega$ are *recursively homeomorphic* if there exist one-to-one, onto, partial recursive functionals $\Phi : P \to Q$ and $\Phi^{-1} : Q \to P$.

Theorem 2.15. *For each $P \subseteq \omega^\omega$ there exists $P^* \subseteq 2^\omega$ such that P is recursively homeomorphic to P^*. It follows that $P \equiv_s P^*$, hence $P \equiv_w P^*$.*

Proof. Identifying subsets of ω with their characteristic functions in 2^ω, let $P^* = \{\{2^m 3^n \mid f(m) = n\} \mid f \in P\}$. Compare Theorem 1.11. $\square$

Remark 2.16. In view of Theorem 2.15, subsets of 2^ω and subsets of ω^ω are identical as to their weak and strong degrees. Compare Remark 1.12. Thus we may write

$$\mathcal{D}_w \;=\; \{\deg_w(P) \mid P \subseteq 2^\omega\}$$

and similarly for $\mathcal{D}_s$.

Remark 2.17. On the other hand, the spaces 2^ω and ω^ω are different in some ways. For instance, 2^ω is compact while ω^ω is not compact. Furthermore, the differences will be important to us. In particular, we shall see that the lattice of weak degrees of nonempty Π^0_1 subsets of 2^ω has a top element, but this is not the case for ω^ω. See Section 4 below.

Remark 2.18. Our main object of study will be the lattice $\mathcal{P}_w$ of weak degrees of nonempty Π^0_1 subsets of the Cantor space, 2^ω. See Definition 3.7 below. We mention Π^0_1 subsets of ω^ω for technical reasons only.

Remark 2.19. Sorbi [43] gives a general survey of the lattices $\mathcal{D}_w$ and $\mathcal{D}_s$ of all weak and strong degrees. However, Sorbi does not discuss the sublattices $\mathcal{P}_w$ and $\mathcal{P}_s$ of weak and strong degrees of nonempty Π^0_1 subsets of 2^ω. The explicit study of $\mathcal{P}_w$ and $\mathcal{P}_s$ began only recently, in 1999, with Simpson [38].

3 Trees and Π^0_1 Sets

Definition 3.1. A predicate $R \subseteq \omega^\omega \times \omega$ is said to be *recursive* if its characteristic function $\chi_R : \omega^\omega \times \omega \to \{0, 1\}$, given by

$$\chi_R(f, n) = \begin{cases} 1 \text{ if } R(f, n), \\ 0 \text{ if } \neg\, R(f, n), \end{cases}$$

is a recursive functional.

Definition 3.2. A set $P \subseteq \omega^\omega$ is said to be Π^0_1 if there is a recursive predicate $R \subseteq \omega^\omega \times \omega$ such that

$$P = \{f \in \omega^\omega \mid \forall n\, R(f, n)\}.$$

Example 3.3. Let T be a consistent, recursively axiomatizable theory. As in Example 2.5, let P_T be the set of (characteristic functions of) completions of T. It is easy to see that P_T is a nonempty Π^0_1 subset of 2^ω. Thus weak and strong reducibility can be used to compare such theories.

In more detail, let $B(T)$ be the *Lindenbaum algebra* of T, i.e., the Boolean algebra of sentences in the vocabulary of T modulo provable equivalence over T. Thus $B(T)$ is a *recursively presented Boolean algebra*, i.e., the quotient of a free recursive Boolean algebra modulo a recursively enumerable ideal. It can be shown that P_T is (recursively homeomorphic to) the Stone space of $B(T)$. Thus we have an effective version of Stone Duality, with nonempty Π^0_1 subsets of 2^ω as the Stone spaces. If T_1 and T_2 are two such theories, then recursive homomorphisms

$$h : B(T_1) \to B(T_2)$$

are in canonical one-to-one correspondence with recursive functionals

$$\Phi : P_{T_2} \to P_{T_1}.$$

In particular, $B(T_1)$ is recursively isomorphic to $B(T_2)$ if and only if P_{T_1} is recursively homeomorphic to P_{T_2}.

Remark 3.4. Conversely, one can show that every nonempty Π^0_1 subset of 2^ω is recursively homeomorphic to P_T for some consistent, recursively axiomatizable theory T. See Theorem 3.18 and Remark 3.19 below.

Remark 3.5. Many other interesting examples of Π^0_1 subsets of 2^ω arise from logic, algebra, analysis, geometry, combinatorics, computability theory, etc. There is a large literature on this subject. See for instance Cenzer/Remmel [8] and Simpson [39, Chapter IV and Section VIII.2].

Remark 3.6. If P and Q are Π^0_1 subsets of 2^ω, then $P \times Q$, $P \cup Q$, $P + Q$ are Π^0_1 subsets of 2^ω. Thus the weak degrees of nonempty Π^0_1 subsets of 2^ω form a sublattice of the lattice of all weak degrees. Similarly for strong degrees, and similarly for subsets of ω^ω.

Definition 3.7. We use $\mathcal{P}_w$ to denote the set of weak degrees of nonempty Π^0_1 subsets of 2^ω. Thus $\mathcal{P}_w$ is a countable sublattice of $\mathcal{D}_w$. Similarly $\mathcal{P}_s$, the set of strong degrees of nonempty Π^0_1 subsets of 2^ω, is a countable sublattice of $\mathcal{D}_s$.

Remark 3.8. The countable distributive lattices $\mathcal{P}_w$ and $\mathcal{P}_s$ are known to have a rich structure.

Binns/Simpson [4, 6] have shown that every countable distributive lattice is lattice embeddable in $\mathcal{P}_w$. A similar conjecture for $\mathcal{P}_s$ remains open, although partial results in this direction are known. A special case is Corollary 9.4 below. Binns [4, 5] has shown that for every $\mathbf{b} > \mathbf{0}$ in $\mathcal{P}_w$ there exist $\mathbf{b}_1, \mathbf{b}_2 < \mathbf{b}$ in $\mathcal{P}_w$ such that $\mathbf{b} = \sup(\mathbf{b}_1, \mathbf{b}_2)$, and similarly for $\mathcal{P}_s$. Cenzer/Hinman [7] have shown that, for all $\mathbf{a}, \mathbf{b} \in \mathcal{P}_s$ such that $\mathbf{a} < \mathbf{b}$, there exists $\mathbf{c} \in \mathcal{P}_s$ such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$. A similar conjecture for $\mathcal{P}_w$ remains open. Binns [4, 5] has improved the result of Cenzer/Hinman [7] by showing that, for all $\mathbf{a}, \mathbf{b} \in \mathcal{P}_s$ such that $\mathbf{a} < \mathbf{b}$, there exist $\mathbf{b}_1, \mathbf{b}_2 < \mathbf{b}$ in $\mathcal{P}_s$ such that $\mathbf{a} < \mathbf{b}_1, \mathbf{b}_2$ and $\mathbf{b} = \sup(\mathbf{b}_1, \mathbf{b}_2)$.

These recent results for $\mathcal{P}_w$ and $\mathcal{P}_s$ are proved by means of priority arguments. They invite comparison with the older, known results for r. e. Turing degrees in Sacks [31] and Soare [42].

Remark 3.9. It is known that $\mathcal{P}_s$ behaves somewhat differently from $\mathcal{P}_w$. To bring out one of the differences, let P, Q be Π^0_1 subsets of 2^ω with $P \leq_s Q$. Thus we have a recursive functional $\Phi : Q \to P$. Using compactness of 2^ω, we can find a total recursive functional $\widehat{\Phi} : 2^\omega \to 2^\omega$ which extends Φ, i.e., $\Phi =$ the restriction of $\widehat{\Phi}$ to Q. Thus, for every $Y \in Q$, $\Phi(Y)$ is not only $\leq_T Y$ but also $\leq_{tt} Y$, where $\leq_{tt}$ denotes truth table reducibility. A general discussion of truth table reducibility is in Rogers [30]. See also Simpson [35, Section 3] and the proof of Lemma 6.5 below. Thus we have an analogy

weak reducibility / strong reducibility $\quad =$

Turing reducibility / truth table reducibility.

In addition, it is known that every nonzero weak degree in $\mathcal{P}_w$ includes infinitely many distinct strong degrees in $\mathcal{P}_s$. See Simpson/Slaman [41].

We now present a useful characterization of Π^0_1 sets, in terms of trees.

Definition 3.10. We use Seq to denote the set of finite sequences of natural numbers. We use $\rho, \sigma, \tau, \ldots$ to denote members of Seq. The length of $\sigma \in$ Seq is denoted $\mathrm{lh}(\sigma)$. The concatenation σ followed by τ is denoted $\sigma^\frown\tau$. Thus $\mathrm{lh}(\sigma^\frown\tau) = \mathrm{lh}(\sigma) + \mathrm{lh}(\tau)$. Given $f \in \omega^\omega$ and $n \in \omega$, we put

$$f[n] = \langle f(0), f(1), \ldots, f(n-1) \rangle.$$

Thus $f[n] \in$ Seq and $\mathrm{lh}(f[n]) = n$. We write $\sigma \subset f$ to mean that $\sigma = f[n]$ for some n. Given $\tau \in$ Seq and $n \leq \mathrm{lh}(\tau)$, we put

$$\tau[n] = \langle \tau(0), \tau(1), \ldots, \tau(n-1) \rangle.$$

We write $\sigma \subseteq \tau$ to mean that $\sigma = \tau[n]$ for some $n \leq \mathrm{lh}(\tau)$.

Definition 3.11. For $e, m, n, t \in \omega$ and $\sigma \in$ Seq, we write

$$\{e\}^\sigma_t(m) = n$$

to mean that $\{e\}^f(m) = n$ for some (or any) $f \in \omega^\omega$ such that $f \supset \sigma$, via a Turing computation which halts in at most t steps and uses only oracle information from σ. Compare Definition 1.3.

Lemma 3.12. *We have:*

1. *If* $\{e\}^{f[s]}_t(m) = n$ *and* $t' \geq t$ *and* $s' \geq s$*, then* $\{e\}^{f[s']}_{t'}(m) = n$.
2. $\{e\}^f(m) = n$ *if and only if* $\{e\}^{f[s]}_t(m) = n$ *for some* $s, t \in \omega$.
3. $\{e\}^f(m) = n$ *if and only if* $\{e\}^{f[s]}_s(m) = n$ *for some* $s \in \omega$.
4. *The relations* $\{e\}^\sigma_t(m) = n$ *and* $\{e\}^\sigma_t(m)\downarrow$ *are recursive.*

Proof. Straightforward. □

Definition 3.13. A *tree* is a set $T \subseteq$ Seq such that, for all $\tau \in T$ and all $n < \mathrm{lh}(\tau)$, $\tau[n] \in T$. A *path* through T is a function $f \in \omega^\omega$ such that, for all $n \in \omega$, $f[n] \in T$.

Theorem 3.14. *A set $P \subseteq \omega^\omega$ is Π^0_1 if and only if there exists a recursive tree $T \subseteq \mathrm{Seq}$ such that*

$$P = \{f \in \omega^\omega \mid f \text{ is a path through } T\}.$$

Proof. Clearly the set of paths through a recursive tree is Π^0_1. Conversely, given a Π^0_1 set $P = \{f \mid \forall n\, R(f,n)\}$, let e be an index of the partial recursive functional $\Phi(f,m) \simeq$ least n such that $\neg R(f,n)$. Then $P = \{f \mid \Phi(f,0)\uparrow\} = \{f \mid \{e\}^f(0)\uparrow\}$. Putting $T = \{\sigma \mid \{e\}^\sigma_{\mathrm{lh}(\sigma)}(0)\uparrow\}$ we see that T is a recursive tree and $P = \{\text{paths through } T\}$. □

Definition 3.15. We use Seq_2 to denote the set of finite sequences of 0's and 1's. Since $\mathrm{Seq}_2 \subseteq \mathrm{Seq}$, the notations introduced in Definitions 3.10 and 3.11 apply.

Definition 3.16. A tree T is said to be *bounded* if for each $n \in \omega$ there are only finitely many $\sigma \in T$ such that $\mathrm{lh}(\sigma) = n$. Note that Seq_2 is a bounded tree, while Seq is an unbounded tree.

Corollary 3.17. *A set $P \subseteq 2^\omega$ is Π^0_1 if and only if there exists a recursive tree $T \subseteq \mathrm{Seq}_2$ such that*

$$P = \{X \in 2^\omega \mid X \text{ is a path through } T\}.$$

Moreover, P is nonempty if and only if T is infinite.

Proof. The first assertion is a special case of Theorem 3.14. The second assertion follows from compactness of 2^ω, in the form of König's Lemma. Namely, a bounded tree has a path if and only if it is infinite. □

We now use Corollary 3.17 to obtain the converse of Example 3.3.

Theorem 3.18.

1. *If T is a consistent, recursively axiomatizable theory, then P_T, the set of completions of T, is a nonempty Π^0_1 subset of 2^ω.*
2. *Conversely, if P is a nonempty Π^0_1 subset of 2^ω, we can find a consistent, recursively axiomatizable theory, T, such that P is recursively homeomorphic to P_T.*

Proof. Part 1 has already been noted in Example 3.3. For part 2, let S be an infinite recursive tree such that $P = \{\text{paths through } S\}$. We use S to construct a theory T in the propositional calculus with atoms A_n, $n \in \omega$. Writing $A^1 = A$ and $A^0 = \neg A$, the axioms of T are all sentences of the form $\neg(A_0^{i_0} \wedge A_1^{i_1} \wedge \cdots \wedge A_k^{i_k})$ where $\langle i_0, i_1, \ldots, i_k\rangle \in \mathrm{Seq}_2 \setminus S$. Then P is recursively homeomorphic to P_T, via $X \mapsto$ the completion of T with axioms $A_n^{X(n)}$, $n \in \omega$. □

Remark 3.19. Surprisingly, it is known that every Π^0_1 subset of 2^ω is recursively homeomorphic to P_T for some finitely axiomatizable theory T in the predicate calculus. See Peretyatkin [29].

We now discuss Π^0_1 subsets of ω^ω and compare them to Π^0_1 subsets of 2^ω.

Definition 3.20. A set $P \subseteq \omega^\omega$ is said to be *recursively bounded* if there exists a recursive function $g \in \omega^\omega$ such that for all $f \in P$, $f(n) < g(n)$ for all n.

Remark 3.21. Clearly any subset of 2^ω is recursively bounded, viz., by the constant function $\lambda n.2$. The next theorem implies that, up to recursive homeomorphism, the study of recursively bounded Π^0_1 subsets of ω^ω is equivalent to the study of Π^0_1 subsets of 2^ω.

Theorem 3.22. *For each recursively bounded Π^0_1 set $P \subseteq \omega^\omega$, we can find a Π^0_1 set $P^* \subseteq 2^\omega$ such that P is recursively homeomorphic to P^*. It follows that $P \equiv_s P^*$, hence $P \equiv_w P^*$.*

Proof. Define P^* as in the proof of Theorem 2.15. It is straightforward to show that, if P is Π^0_1 and recursively bounded, then P^* is Π^0_1. $\square$

Remark 3.23. Conversely, if $P \subseteq 2^\omega$ is Π^0_1, then for any recursive functional $\Phi : P \to \omega^\omega$, the range $\{\Phi(f) \mid f \in P\}$ is Π^0_1 and recursively bounded. This is a consequence of compactness of 2^ω.

Remark 3.24. By Theorem 3.22, the weak degrees of recursively bounded Π^0_1 subsets of ω^ω belong to $\mathcal{P}_w$, and similarly for strong degrees. On the other hand, there are plenty of nonempty Π^0_1 subsets of ω^ω whose weak degrees do not belong to $\mathcal{P}_w$.

Example 3.25. It is known from hyperarithmetical theory (see Sacks [32, Part A] or Simpson [39, Section VIII.3]) that for any hyperarithmetical $X \in 2^\omega$ there exists a hyperarithmetical $g \in \omega^\omega$ such that $X \leq_T g$ and the singleton set $\{g\} \subseteq \omega^\omega$ is Π^0_1. If g is not recursive, the GKT Basis Theorem (see Simpson [39, Section VIII.2]) implies that $\deg_w(\{g\}) \not\leq_w P$ for any nonempty Π^0_1 set $P \subseteq 2^\omega$.

Example 3.26. Another interesting Π^0_1 subset of ω^ω is

$$\mathrm{DNR} = \{f \in \omega^\omega \mid \forall n\, f(n) \neq \{n\}(n)\},$$

i.e., the set of $f : \omega \to \omega$ which are *diagonally non-recursive*. We shall comment more on this later. See Corollary 7.3 and Remark 7.5 below.

4 Weak and Strong Completeness

Definition 4.1. A nonempty Π^0_1 set $P \subseteq 2^\omega$ is said to be *weakly complete*, or *Muchnik complete*, if every nonempty Π^0_1 subset of 2^ω is weakly reducible to P.

Definition 4.2. A nonempty Π^0_1 set $P \subseteq 2^\omega$ is said to be *strongly complete*, or *Medvedev complete*, if every nonempty Π^0_1 subset of 2^ω is strongly reducible to P.

Remark 4.3. We use $\mathbf{1}$ to denote the weak degree of any nonempty Π^0_1 subset of 2^ω which is weakly complete. Thus $\mathbf{1}$ is the top element of $\mathcal{P}_w$. Similarly for strong degrees and $\mathcal{P}_s$.

Example 4.4. The following Π^0_1 subsets of 2^ω are known to be strongly complete, hence weakly complete.

1. $P = \{\text{completions of } \mathsf{PA}\}$. Instead of PA we could use any effectively axiomatizable, effectively essentially undecidable theory. This is related to the Gödel/Rosser Theorem. See also Scott/Tennenbaum [33].

2. $P = \{f \in 2^\omega \mid f \text{ separates } A \text{ and } B\}$, where $A = \{n \mid \{n\}(n) \simeq 0\}$ and $B = \{n \mid \{n\}(n) \simeq 1\}$. See Jockusch/Soare [20].

3. We can also give an explicit, recursion-theoretic construction of a Π^0_1 set P with the desired property. Namely, $P = \prod_{e=0}^{\infty} P_e^+$ where P_e^+ is the nonempty Π^0_1 subset of 2^ω indexed by e. See Simpson [35, Lemma 3.3].

Theorem 4.5 (Simpson 2000). *Any two strongly complete Π^0_1 subsets of 2^ω are recursively homeomorphic.*

Proof. The proof is by an effective back-and-forth argument, using the Recursion Theorem. See Simpson [35, Section 3]. It is analogous to the proof of Myhill's result that any two creative, recursively enumerable subsets of ω are recursively isomorphic. Myhill's result is expounded in Rogers [30]. □

Corollary 4.6. *A nonempty Π^0_1 subset of 2^ω is strongly complete if and only if it is recursively homeomorphic to the set of completions of* PA*.*

The proof of Theorem 4.5 also gives the following.

Corollary 4.7. *Let P and Q be nonempty Π^0_1 subsets of 2^ω. If P is strongly complete, then there is a recursive functional $\Phi : P \to Q$ which maps P* onto *Q, i.e., $Q = \{\Phi(f) \mid f \in P\}$.*

Proof. See Simpson [35, Section 3]. □

The following example shows that strong completeness is not the same as weak completeness.

Example 4.8 (Jockusch 1989). For $k \geq 2$ let DNR_k be the set of functions $f : \omega \to \{1, \ldots, k\}$ which are DNR. It is easy to see that the sets DNR_k, $k = 2, 3, \ldots$, are Π^0_1 and recursively bounded, and that DNR_2 is strongly complete. Jockusch [19] has shown that the the sets DNR_k, $k = 2, 3, \ldots$ are weakly complete but of different strong degrees. Thus we have $\mathrm{DNR}_2 \equiv_w \mathrm{DNR}_3 \equiv_w \cdots$ yet $\mathrm{DNR}_2 >_s \mathrm{DNR}_3 >_s \cdots$.

An interesting relationship between weak and strong reducibility is given by the following theorem.

Theorem 4.9 (Simpson 2001). *Let $P, Q \subseteq 2^\omega$ be nonempty Π^0_1 sets. If $P \leq_w Q$, then there exists a nonempty Π^0_1 set $Q' \subseteq Q$ such that $P \leq_s Q'$.*

Proof. We shall prove this later, as a consequence of the Almost Recursive Basis Theorem. See Theorem 6.6. □

Corollary 4.10. *If $Q \subseteq 2^\omega$ is Π^0_1 and weakly complete, then there is a Π^0_1 set $Q' \subseteq Q$ such that Q' is strongly complete.*

Definition 4.11. $P, Q \subseteq \omega^\omega$ are said to be *Turing degree isomorphic* if there exists a Turing-degree-preserving one-to-one correspondence between P and Q. Clearly recursive homeomorphism implies Turing degree isomorphism.

Theorem 4.12 (Simpson 2001). *Any two weakly complete Π^0_1 subsets of 2^ω are Turing degree isomorphic.*

Proof. This follows easily from Theorem 4.5 and Corollary 4.10. □

Corollary 4.13. *A nonempty Π^0_1 subset of 2^ω is weakly complete if and only if it is Turing degree isomorphic to the set of completions of* PA.

Corollary 4.14. *Any two nonempty Π^0_1 subsets of $\bigcup_{k=0}^\infty \mathrm{DNR}_k$ are Turing degree isomorphic.*

Corollary 4.15. *If P is weakly complete, then the set of Turing degrees of members of P is upward closed.*

Proof. Let P be weakly complete. Put $Q = P \times 2^\omega$. Clearly Q is weakly complete, and the set of Turing degrees of members of Q is upward closed. By Theorem 4.12, P and Q are Turing degree isomorphic. □

Corollary 4.16 (Solovay). *The set of Turing degrees of completions of* PA *is upward closed.*

5 1-Randomness

In this section we present an explicit, natural example of a weak degree in $\mathcal{P}_w$ which is strictly between $\mathbf{0}$ and $\mathbf{1}$. Our example is based on Martin-Løf's theory of randomness.

We use the "fair coin" probability measure on 2^ω. Thus for all $n \in \omega$ we have

$$\mu(\{X \in 2^\omega \mid X(n) = 0\}) = \mu(\{X \in 2^\omega \mid X(n) = 1\}) = 1/2.$$

Definition 5.1. An *effective null G_δ* is a set $S \subseteq 2^\omega$ of the form $S = \bigcap_{n=0}^\infty U_n$ where U_n, $n \in \omega$, is a recursively indexed sequence of Σ^0_1 sets such that $\mu(U_n) < 1/2^n$ for all n.

Definition 5.2. $X \in 2^\omega$ is 1-*random* if $X \notin S$ for all effective null G_δ sets S. The set of 1-random $X \in 2^\omega$ is denoted R_1. Clearly $\mu(R_1) = 1$.

Theorem 5.3 (Martin-Løf 1966). *The union of all effective null G_δ sets is an effective null G_δ set.*

Proof. This result is due to Martin-Løf [26]. The proof is by a diagonal argument. See also Kučera [22]. □

Corollary 5.4. *$2^\omega \setminus R_1$ is an effective null G_δ set. Hence R_1 is Σ^0_2.*

Corollary 5.5. *$R_1 = \bigcup_{n=0}^\infty P_n$ where P_n, $n \in \omega$, is a sequence of Π^0_1 sets.*

Theorem 5.6. *Let $Q \subseteq 2^\omega$ be Π^0_1 of measure 0. Then Q is an effective null G_δ set.*

Proof. Straightforward. □

Corollary 5.7. *Let $Q \subseteq 2^\omega$ be Π^0_1. We have $\mu(Q) > 0$ if and only if $Q \cap R_1 \neq \emptyset$. In this case we actually have $Q \cap R_1 \supseteq P \neq \emptyset$, where P is Π^0_1 and $\mu(P) > 0$.*

Theorem 5.8 (Kučera 1985). *Let $Q \subseteq 2^\omega$ be Π^0_1 with $\mu(Q) > 0$. Then for all 1-random $X \in 2^\omega$ we have that $X^{(k)} \in Q$ for some $k \in \omega$. Here $X^{(k)}(n) = X(k+n)$ for all $n \in \omega$.*

Proof. See Kučera [22]. Let T be a recursive tree such that Q is the set of paths through T. Let $\widetilde{T}$ be the set of all $\tau^\frown\langle i\rangle \in \mathrm{Seq}_2$ such that $\tau \in T$ and $\tau^\frown\langle i\rangle \notin T$. Let Q^2 be the set of paths through the tree $T^2 = T \cup \{\sigma^\frown\tau \mid \sigma \in \widetilde{T}, \tau \in T\}$. Note that Q^2 is Π^0_1 and $\mu(Q^2) = 1 - (1-\mu(Q))^2$. Define Q^n similarly for all $n \geq 1$. Since Q^n is Π^0_1 and $\mu(Q^n) = 1-(1-\mu(Q))^n$, we have that $2^\omega \setminus \bigcup_{n=1}^\infty Q^n$ is an effective null G_δ set. Hence $X \in Q^n$ for some n. It follows that $X^{(k)} \in Q$ for some k. □

Corollary 5.9. *Let $Q \subseteq 2^\omega$ be Π^0_1 with $\mu(Q) > 0$. Then $Q \leq_w R_1$.*

Corollary 5.10. *Let Q be a nonempty Π^0_1 subset of R_1. Then $Q \equiv_w R_1$.*

Corollary 5.11. *Among all weak degrees of Π^0_1 sets $Q \subseteq 2^\omega$ with $\mu(Q) > 0$, there is a largest one, and it is the same as the weak degree of R_1. Call this weak degree $\mathbf{r}_1$.*

Theorem 5.12. *Let $A, B \subseteq \omega$ be recursively inseparable. Then*

$$\mu(\{X \in 2^\omega \mid \exists Y \leq_T X\,(Y \text{ separates } A, B)\}) = 0.$$

Proof. Not difficult. See Jockusch/Soare [20]. □

Corollary 5.13. *The weak degree $\mathbf{r}_1 = \deg_w(R_1) \in \mathcal{P}_w$ of Corollary 5.11 is not weakly complete. We have $\mathbf{0} < \mathbf{r}_1 < \mathbf{1}$.*

Remark 5.14. More generally, for all weak degrees $\mathbf{a} \in \mathcal{D}_w$, if $\sup(\mathbf{a}, \mathbf{r_1}) \geq \mathbf{1}$ then $\mathbf{a} \geq \mathbf{1}$. This result is due to Simpson [40].

Remark 5.15. The weak degree $\mathbf{r}_1$ is the first explicit, natural example of a weak degree in $\mathcal{P}_w$ strictly between $\mathbf{0}$ and $\mathbf{1}$. This is especially interesting because no explicit, natural examples of r. e. Turing degrees strictly between $\mathbf{0}$ and $\mathbf{0}'$ are known. See Simpson [38].

6 The Almost Recursive Basis Theorem

Definition 6.1. X is *almost recursive* (a.k.a., *hyperimmune-free*) if, for each function $f : \omega \to \omega$ recursive in X, there exists a recursive function $g : \omega \to \omega$ such that $f(m) < g(m)$ for all $m \in \omega$.

The following theorem is from Jockusch/Soare [20]. We call it the Almost Recursive Basis Theorem.

Theorem 6.2. *Let P be a nonempty Π^0_1 subset of 2^ω. Then there exists $X \in P$ such that X is almost recursive.*

Proof. Define a sequence of nonempty Π^0_1 sets $P = P_0 \supseteq P_1 \supseteq \cdots \supseteq P_n \supseteq \cdots$ as follows. Put $P_0 = P$. If $\exists m\,(\exists X \in P_n)\,\{n\}^X(m)\uparrow$, fix such an m and put $P_{n+1} = \{X \in P_n \mid \{n\}^X(m)\uparrow\}$. Otherwise, put $P_{n+1} = P_n$. Clearly there is a unique $X \in \bigcap_{n=0}^\infty P_n$. By Remark 3.23, X is almost recursive. □

Corollary 6.3. *There exists a completion of* PA *which is almost recursive.*

Corollary 6.4. *There exists a 1-random $X \in 2^\omega$ which is almost recursive.*

Lemma 6.5. *Suppose X is almost recursive and $X \geq_T Y$. Then Y is truth table reducible to X. In particular, there exists a total recursive functional $\Phi : 2^\omega \to 2^\omega$ such that $\Phi(X) = Y$.*

Proof. Let e be such that $Y = \{e\}^X$. Define $f : \omega \to \omega$ by $f(m) =$ the least s such that $\{e\}^{X[s]}_s(m)\downarrow$. Clearly $f \leq_T X$. Let $g : \omega \to \omega$ be recursive such that $f(m) \leq g(m)$ for all n. Define a truth table functional $\Phi : 2^\omega \to 2^\omega$ by putting $\Phi(Z)(m) = \{e\}^{Z[g(m)]}_{g(m)}(m)$ if this is defined, and $\Phi(Z)(m) = 0$ otherwise. Clearly $\Phi(X) = Y$. □

The following theorem from Simpson [40] provides an interesting relationship between $\leq_w$ and $\leq_s$.

Theorem 6.6. *Let $P, Q \subseteq 2^\omega$ be nonempty Π^0_1 sets. If $P \leq_w Q$, then there is a nonempty Π^0_1 set $Q' \subseteq Q$ such that $P \leq_s Q'$.*

Proof. Assume $P \leq_w Q$. By Theorem 6.2 let $Y \in Q$ be almost recursive. Let $X \in P$ be such that $X \leq_T Y$. By Lemma 6.5 let $\Phi : 2^\omega \to 2^\omega$ be a truth table functional such that $\Phi(Y) = X$. Put $Q' = Q \cap \Phi^{-1}(P)$. Then Q' is a nonempty Π^0_1 subset of Q, and $P \leq_s Q'$ via Φ. □

Corollary 6.7. *Let X be 1-random and almost recursive. Then there is no completion of* PA *which is $\leq_T X$.*

Proof. Otherwise, by the proof of Theorem 6.6, there would be a strongly complete Π^0_1 set $Q \subseteq 2^\omega$ with $\mu(Q) > 0$. This would contradict Theorem 5.12. □

7 The $\Sigma^0_3 \to \Pi^0_1$ Embedding Theorem

The next theorem, due to Simpson [40], tells us that the weak degrees of many naturally occurring mass problems belong to $\mathcal{P}_w$, even when they do not naturally occur as recursively bounded Π^0_1 sets.

Definition 7.1. A set $S \subseteq \omega^\omega$ is said to be Σ^0_3 if there exists a recursive predicate $R \subseteq \omega^\omega \times \omega^3$ such that

$$S = \{f \in \omega^\omega \mid \exists n_1 \, \forall n_2 \, \exists n_3 \, R(f, n_1, n_2, n_3)\}.$$

A set $P \subseteq \omega^\omega$ is said to be Π^0_3 if its complement $\omega^\omega \setminus P$ is Σ^0_3. One defines Σ^0_k and Π^0_k similarly for all $k \geq 1$. See Rogers [30, Chapter 15].

Theorem 7.2. *If $S \subseteq \omega^\omega$ is Σ^0_3, then for all nonempty Π^0_1 sets $P \subseteq 2^\omega$ we can find a Π^0_1 set $Q \subseteq 2^\omega$ such that $Q \equiv_w P \cup S$.*

Proof. First use a Skolem function technique to reduce to the case where S is a Π^0_1 subset of ω^ω. Namely, replace S by the set of all $\langle k \rangle^\frown (f \oplus g) \in \omega^\omega$ such that $\forall m \, (g(m) =$ the least n such that $R(f, k, m, n))$. Clearly this set is $\equiv_w S$ and Π^0_1. After that, let T_S be a recursive subtree of Seq such that S is the set of paths through T_S. Let T_P be a recursive subtree of Seq_2 such that P is the set of paths through T_P. We may assume that, for all $\tau \in T_S$ and $n < \text{lh}(\tau)$, $\tau(n) \geq 2$. Define T_Q to be the set of sequences $\rho \in$ Seq of the form

$$\sigma_0{}^\frown\langle n_0 \rangle^\frown \sigma_1{}^\frown \langle n_1 \rangle^\frown \cdots {}^\frown \langle n_{k-1} \rangle^\frown \sigma_k$$

where $\langle n_0, n_1, \ldots, n_{k-1} \rangle \in T_S$, $\sigma_0, \sigma_1, \ldots, \sigma_k \in T_P$, and $\rho(m) \leq m + 2$ for all $m < \text{lh}(\rho)$. Thus T_Q is a recursive subtree of Seq. Let $Q \subseteq \omega^\omega$ be the set of paths through T_Q. It is not hard to see that $Q \equiv_w P \cup S$. Note that Q is Π^0_1 and recursively bounded. Hence by Theorem 3.22 there is a Π^0_1 set $Q^* \subseteq 2^\omega$ which is recursively homeomorphic to Q. □

Corollary 7.3. *There is a Π^0_1 set $D \subseteq 2^\omega$ such that $D \equiv_w$ DNR.*

Proof. Apply Theorem 7.2 with $P = \text{DNR}_2$ and $S = \text{DNR}$. □

Remark 7.4. Put $\mathbf{d} = \deg_w(D) = \deg_w(\text{DNR})$. By Kumabe [21] (see also Ambos-Spies/Kjos-Hanssen/Lempp/Slaman [2]) we have

$$\mathbf{0} < \mathbf{d} < \mathbf{r}_1 < \mathbf{1}.$$

The weak degrees $\mathbf{1}$, $\mathbf{r}_1$, and $\mathbf{d}$ correspond to the system WKL_0 and two of its subsystems which have arisen in the foundations of mathematics. See respectively Simpson [39], Yu/Simpson [45], and Giusto/Simpson [18].

Remark 7.5. Jockusch [19] has shown that the following mass problems are pairwise Turing degree isomorphic, hence weakly equivalent.

1. $\text{DNR} = \{f \in \omega^\omega \mid f$ is diagonally non-recursive, i.e., $\forall e \, f(e) \not\simeq \{e\}(e)\}$.

2. FPF $= \{f \in \omega^\omega \mid f$ is fixed point free, i.e., $\forall e\, \exists m\, \{f(e)\}(m) \not\simeq \{e\}(m)\}$.
3. EI $= \{A \subseteq \omega \mid A$ is effectively immune$\}$.
   This means that A is infinite and, given an index of an r. e. set $C \subseteq A$, we can effectively find a finite upper bound for the cardinality of C.
4. EBI $= \{A \subseteq \omega \mid A$ is effectively bi-immune$\}$.
   This means that both A and $\omega \setminus A$ are effectively immune.

Definition 7.6. A member of 2^ω is said to be 2-*random* if it is 1-random relative to $\mathbf{0}'$, the Turing degree of the Halting Problem. The set of 2-random $X \in 2^\omega$ is denoted R_2. We write $\mathbf{r}_2 = \deg_w(R_2)$.

Corollary 7.7. *There is a Π^0_1 set $R_2^* \subseteq 2^\omega$ such that $R_2^* \equiv_w R_2 \cup P$, where $P = \{$completions of $\mathsf{PA}\}$. Put $\mathbf{r}_2^* = \inf(\mathbf{r}_2, \mathbf{1}) = \deg_w(R_2^*)$.*

Proof. Relativizing Corollary 5.4 we see that R_2 is a Σ^0_3 subset of 2^ω. Our result then follows by Theorem 7.2. □

Theorem 7.8. *If X is 2-random, then X is not almost recursive.*

Proof. Martin [24] has shown that $\mu(\{X \in 2^\omega \mid X$ is almost recursive$\}) = 0$. Our theorem follows from an analysis of Martin's proof. See also the exposition of Martin's result in Dobrinen/Simpson [11]. □

Theorem 7.9. *We have $\mathbf{0} < \mathbf{d} < \mathbf{r}_1 < \mathbf{r}_2^* < \mathbf{1}$.*

Proof. From Remark 7.4 we have $\mathbf{0} < \mathbf{d} < \mathbf{r}_1$, and obviously $\mathbf{r}_1 \leq \mathbf{r}_2^* \leq \mathbf{1}$. Theorem 5.12 implies that $\mathbf{r}_2^* < \mathbf{1}$. The fact that $\mathbf{r}_1 < \mathbf{r}_2^*$ follows from Corollaries 6.4 and 6.7 and Theorem 7.8. □

Remark 7.10. Additional examples of naturally occurring mass problems whose weak degrees belong to $\mathcal{P}_w$ are in Simpson [40].

8 Embedding the R. E. Turing Degrees

Recall that $\mathcal{R}_T$ is the upper semilattice of Turing degrees of recursively enumerable subsets of ω, and $\mathcal{P}_w$ ($\mathcal{P}_s$) is the lattice of weak (strong) degrees of nonempty Π^0_1 subsets of 2^ω. See Remark 1.17 and Definition 3.7.

In this section we use the $\Sigma^0_3 \to \Pi^0_1$ Embedding Theorem 7.2 to embed $\mathcal{R}_T$ into $\mathcal{P}_w$. We do not know whether there exists an embedding of $\mathcal{R}_T$ into $\mathcal{P}_s$.

Theorem 8.1. *Let $A \in 2^\omega$ be Δ^0_2, i.e., $\deg_T(A) \leq_T \mathbf{0}'$. Then there is a Π^0_1 set $P_A \subseteq 2^\omega$ such that $P_A \equiv_w P \cup \{A\}$, where $P = \{$completions of $\mathsf{PA}\}$. We have $P_{A \oplus B} \equiv_w P_A \times P_B$.*

Proof. The first statement follows from Theorem 7.2 since the singleton set $\{A\}$ is Π^0_2. The second statement is straightforward. □

Theorem 8.2 (Arslanov Completeness Criterion). *Let $A \subseteq \omega$ be recursively enumerable. If $f \in \mathrm{DNR}$ and $f \leq_T A$, then A is Turing complete, i.e., $\deg_T(A) = \mathbf{0}'$.*

Proof. See Soare's book [42, Section V.5]. Note that we are identifying $A \subseteq \omega$ with its characteristic function $\chi_A \in 2^\omega$. □

Theorem 8.3. *Let $A, B \subseteq \omega$ be recursively enumerable. Then $A \leq_T B$ if and only if $P_A \leq_w P_B$.*

Proof. Obviously $A \leq_T B$ implies $P_A \leq_w P_B$. For the converse, recall that P is strongly complete, hence recursively homeomorphic to DNR_2. In particular, for all $X \in P$ there is a DNR function $f \leq_T X$. Assume now that $P_A \leq_w P_B$. In particular we can find $X \in P \cup \{A\}$ such that $X \leq_T B$. If $X \in P$, then by the Arslanov Completeness Criterion, B is Turing complete, hence $A \leq_T B$. If $X \notin P$, then $X = A$, hence again $A \leq_T B$. □

Remark 8.4. Thus our embedding of the r. e. Turing degrees into the weak lattice $\mathcal{P}_w$ is given by $\deg_T(A) \mapsto \deg_w(P \cup \{A\})$, where $P = \{$completions of $\mathsf{PA}\}$. The embedding is one-to-one, order preserving, least upper bound preserving, and carries $\mathbf{0}$ to $\mathbf{0}$ and $\mathbf{0}'$ to $\mathbf{1}$.

Remark 8.5. Instead of $P = \{$completions of $\mathsf{PA}\}$, we could use any nonempty Π^0_1 set $P \subseteq 2^\omega$ such that $\mathrm{DNR} \leq_w P$. Compare Corollary 7.3. Thus, for any $\mathbf{c} \in \mathcal{P}_w$ such that $\mathbf{c} \geq \mathbf{d} = \deg_w(\mathrm{DNR})$, we obtain an embedding of the r. e. Turing degrees into $\{\mathbf{a} \in \mathcal{P}_w \mid \mathbf{0} \leq \mathbf{a} \leq \mathbf{c}\}$. The embbedding is one-to-one, order preserving, least upper bound preserving, and carries $\mathbf{0}$ to $\mathbf{0}$ and $\mathbf{0}'$ to $\mathbf{c}$.

9 A Priority Argument

In this section we sketch the construction of a Π^0_1 set $P \subseteq 2^\omega$ with several interesting properties. The construction uses a priority argument.

Definition 9.1. A Π^0_1 set $P \subseteq 2^\omega$ is said to be *thin* if, for all Π^0_1 sets $Q \subseteq P$, there is a finite set $\sigma_1, \ldots, \sigma_n \in \mathrm{Seq}_2$ such that

$$Q = \{X \in P \mid \sigma_1 \subset X \vee \cdots \vee \sigma_n \subset X\}.$$

This is equivalent to saying that, for all Π^0_1 sets $Q \subseteq P$, $P \setminus Q$ is Π^0_1. See also references [25, 12, 13, 9].

Definition 9.2. A family of Turing degrees $\{\mathbf{a}_i \mid i \in I\}$ is said to be *independent* if for all finite $\{i_0, i_1, \ldots, i_n\} \subseteq I$, $\mathbf{a}_{i_0} \leq \sup(\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_n})$ implies $i_0 \in \{i_1, \ldots, i_n\}$.

Theorem 9.3. *We can construct a Π^0_1 set $P \subseteq 2^\omega$ with the following properties:*

1. *P is thin and of cardinality $2^{\aleph_0}$.*

2. *P has no recursive members.*

3. *The Turing degrees* $\deg_T(X)$*,* $X \in P$*, are independent.*
4. *For all* $X \in P$*, putting* $\mathbf{a} = \deg_T(X)$*, we have* $\mathbf{a}' = \sup(\mathbf{a}, \mathbf{0}')$*.*
   Here $\mathbf{a}'$ *denotes the Turing jump of* $\mathbf{a}$*.*

Furthermore, given a Π^0_1 *set* $Q \subseteq 2^\omega$ *with no recursive members, we can arrange that no member of* P *Turing computes a member of* Q*.*

Sketch of proof. We follow Binns/Simpson [6] building on the techniques of Martin/Pour-El [25] and Jockusch/Soare [20, Theorem 4.7].

By a *treemap* we mean a function $h : \mathrm{Seq}_2 \to \mathrm{Seq}_2$ such that $h(\sigma)^\frown\langle i\rangle \subseteq h(\sigma^\frown\langle i\rangle)$ for all $\sigma \in \mathrm{Seq}_2$, $i \in \{0,1\}$.

Starting with h_0 = the identity map, we construct a recursive sequence of recursive treemaps h_s, $s \in \omega$, which are nested in the sense that for all s and all $\sigma \in \mathrm{Seq}_2$ there exists $\tau \in \mathrm{Seq}_2$ such that $h_{s+1}(\sigma) = h_s(\tau)$. After presenting the recursive construction, we argue that, for all $\sigma \in \mathrm{Seq}_2$, the limit $h(\sigma) = \lim_s h_s(\sigma)$ exists and is finite. If follows that $h = \lim_s h_s$ is a treemap, and we define

$$P = \{X \in 2^\omega \mid \forall n\,(\exists\sigma \text{ of length } n)\, h(\sigma) \subset X\}.$$

Clearly P will be Π^0_1 and of cardinality $2^{\aleph_0}$.

In order to insure that P is thin, we arrange that for all $e \in \omega$ and all $\sigma \in \mathrm{Seq}_2$ of length e, $\{e\}^{h(\sigma)}(0)\downarrow$ "if possible". Then for all $X \in P$ we have

$$H^X = \{e \mid (\exists\sigma \text{ of length } e)\,(\{e\}^{h(\sigma)}_{\mathrm{lh}(h(\sigma))}(0)\downarrow \text{ and } h(\sigma) \subset X)\},$$

so $H^X \leq_T H \oplus X$ and this gives property 4. Now, given a Π^0_1 set $Q \subseteq 2^\omega$, let e be such that $Q = \{X \mid \{e\}^X(0)\uparrow\}$. (See the proof of Theorem 3.14.) Then

$$Q = \{X \in P \mid (\exists\sigma \text{ of length } e)\,(\{e\}^{h(\sigma)}_{\mathrm{lh}(h(\sigma))}(0)\uparrow \text{ and } h(\sigma) \subset X)\},$$

and this gives thinness.

The strategy for property 3 is similar. For example, to insure $X \not\leq_T Y$ for all $X, Y \in P$ with $X \neq Y$, we arrange that for all e and all $\sigma, \tau \in \mathrm{Seq}_2$ of length e with $\sigma \neq \tau$, $\exists m < \mathrm{lh}(h(\sigma))\,(\{e\}^{h(\tau)}_{\mathrm{lh}(h(\tau))}(m)\downarrow\, \neq h(\sigma)(m))$ "if possible".

The final property is obtained by means of a Sacks preservation strategy. See Binns/Simpson [6]. □

Corollary 9.4. *Every finite distributive lattice is lattice embeddable in* $\mathcal{P}_w$ *and in* $\mathcal{P}_s$*.*

Proof. First note that any finite distributive lattice is lattice embeddable in the free distributive lattice on n generators, for sufficiently large n. Now let P be as in Theorem 9.3, and let $P_1, \ldots, P_n$ be nonempty, pairwise disjoint, Π^0_1 subsets of P. In view of property 3, the weak or strong degrees of $P_1, \ldots, P_n$ are independent and hence freely generate a free distributive lattice. Details are in Binns/Simpson [6]. □

Corollary 9.5. *For any* $\mathbf{b} > \mathbf{0}$ *in* $\mathcal{P}_w$ *or* $\mathcal{P}_s$*, every finite distributive lattice is lattice embeddable in the interval* $\mathbf{0} \le \mathbf{a} \le \mathbf{b}$*.*

Proof. Let $Q \subseteq 2^\omega$ be Π^0_1 with $\mathbf{b} = \deg_w(Q)$ or $\deg_s(Q)$ as the case may be. Let P be as in Theorem 9.3 such that no member of P Turing computes a member of Q. Proceed as in the proof of Corollary 9.4, replacing $P_1, \ldots, P_n$ by $P_1 + Q, \ldots, P_n + Q$. □

Remark 9.6. By Theorem 9.3, let P be a nonempty thin Π^0_1 subset of 2^ω with no recursive members. Then P is of measure 0, and in fact, $\deg_w(P)$ is incomparable with $\mathbf{r}_1$. These results are due to Simpson [40].

Remark 9.7. By Theorem 9.3 and Remark 3.19, let T be a consistent, finitely axiomatizable, essentially undecidable theory such that P_T is thin. Then any recursively axiomatizable theory extending T with the same vocabulary as T is finitely axiomatizable. Compare Martin/Pour-El [25].

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