Predicativism:

a reverse-mathematical perspective

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Foundations of mathematics is the study of the basic concepts and logical structure of mathematics as a whole.

Many central problems in foundations of mathematics arise from questions about the notion of "infinity." Let us focus on three foundational doctrines: *finitism*, *predicativism*, and *impredicativism*.

- **Finitism** says that the natural number system $\mathbb{N}$ exists only as a *potential infinity*;

- **Predicativism** says that $\mathbb{N}$ exists as an *actual infinity*;

- **Impredicativism** says that not only $\mathbb{N}$ but also many other actual infinities exist.

These doctrines can be formalized in terms of various axiom systems.

Reverse mathematics is a program of foundationally-inspired research which focuses on questions of the following type.

> Which axioms are needed in order to prove specific theorems in core mathematical subjects?

(Some examples of core mathematical subjects are: analysis, algebra, geometry, topology, differential equations, combinatorics.)

Reverse mathematicians have uncovered definitive answers to many questions of this type. These results have specific implications for doctrines such as finitism, predicativism, and impredicativism.

Reverse mathematics takes place in **the Gödel hierarchy**.
Almost all reverse-mathematical research involves a particular family of formal systems, namely, subsystems of second-order arithmetic.
Five particular subsystems of Z_2 have played a large role:
RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0. These are known as "The Big Five."

The Gödel hierarchy for reverse mathematics

"strong"

- ⋮
- huge cardinal numbers
- ⋮
- ineffable cardinal numbers
- ⋮
- ZFC (Zermelo/Fraenkel set theory)
- ZC (Zermelo set theory)
- simple type theory

"medium"

- Z_2 (second-order arithmetic)
- ⋮
- Π^1_2-CA_0 (Π^1_2 comprehension)
- Π^1_1-CA_0 (Π^1_1 comprehension)
- ATR_0 (arithmetical transfinite recursion)
- ACA_0 (arithmetical comprehension)

"weak"

- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- EFA (elementary function arithmetic)
- bounded arithmetic
- ⋮

The reverse-mathematical perspective:

- **finitism** is embodied by PRA, RCA_0, and WKL_0;
- **predicativism** is embodied by ACA_0 and IR and ATR_0;
- **impredicativism** is embodied by Π^1_1-CA_0 and stronger systems.

Details:

PRA embodies the outer limits of finitistic reasoning (Tait 1981).

RCA_0 and WKL_0 are not finitistic, but they are **finitistically reducible** in the following sense: RCA_0 is Π^0_2-conservative over PRA (Parsons 1970), and WKL_0 is Π^1_1-conservative over RCA_0 (Harrington 1977).

ACA_0 is necessary and sufficient for the development of predicative analysis, following Weyl's monograph *Das Kontinuum*. Proof-theoretically, ACA_0 is much stronger than WKL_0. The first-order part of ACA_0 is $PA = Z_1 =$ first-order arithmetic.

Inspired by Poincaré and Weyl, Feferman 1964 developed a formal system IR which embodies the outer limits of predicative reasoning. Proof-theoretically, IR is much stronger than ACA_0.

ATR_0 is not predicative, but it is **predicatively reducible** in the following sense: ATR_0 is Π^1_1-conservative over IR (FMS 1982).

Reverse mathematics for WKL_0.

WKL_0 is equivalent over RCA_0 to each of the following statements:

1. The Heine/Borel Covering Lemma: Every covering of $[0, 1]$ by a sequence of open intervals has a finite subcovering.
2. Every covering of a compact metric space by a sequence of open sets has a finite subcovering.
3. Every continuous real-valued function on $[0, 1]$ (or on any compact metric space) is bounded (unif. continuous, Riemann integrable).
6. The Maximum Principle: Every continuous r.-v. function on $[0, 1]$ (or on any compact metric space) has (or attains) a supremum.
7. The local existence theorem for solutions of (finite systems of) ordinary differential equations.
8. Gödel Completeness Theorem: Every finite (or countable) consistent set of sentences in the predicate calculus has a countable model.
9. Gödel Compactness Theorem for countable sets of sentences in propositional calculus.

Reverse mathematics for WKL_0, continued.

10. Every countable commutative ring has a prime ideal.
11. Every countable field (of characteristic 0) has a unique algebraic closure.
12. Every countable formally real field is orderable.
13. Every countable formally real field has a (unique) real closure.
14. The Brouwer Fixed Point Theorem.
15. The Schauder Fixed Point Theorem for separable Banach spaces.
16. The Hahn-Banach Theorem for separable Banach spaces.

A remark on Hilbert's Program of finitistic reductionism.

There are many true mathematical statements which fail recursively, in the sense that we can construct "recursive counterexamples." These statements are not provable in RCA_0, because RCA_0 is recursively true. However, many of these statements are provable in WKL_0, and WKL_0 is Π^1_1-conservative over RCA_0, hence Π^0_2-conservative over PRA. Thus we have a partial realization of Hilbert's Program of finitistic reductionism.

Another remark on Hilbert's Program of finitistic reductionism.

Recently Ludovic Patey and Keita Yokoyama solved a fascinating and long-standing open problem.

Namely, they calibrated the proof-theoretical strength of RT(2, 2) = Ramsey's Theorem for 2-colorings of pairs.

For any set X, let $[X]^2$ be the set of all 2-element subsets of X. RT(2, 2) says that for any 2-coloring $[\mathbb{N}]^2 = C_1 \cup C_2$, there exists an infinite set $X \subseteq \mathbb{N}$ such that $[X]^2 \subseteq C_1$ or $[X]^2 \subseteq C_2$.

Patey and Yokoyama showed that $\mathsf{WKL}_0 + \mathsf{RT}(2,2)$ is Π^0_3-conservative over RCA_0, hence Π^0_2-conservative over PRA. Thus we can say that RT(2, 2), like WKL_0, is finitistically reducible à la Hilbert's Program.

It remains open whether $\mathsf{WKL}_0 + \mathsf{RT}(2,2)$ is Π^1_1-conservative over RCA_0.

Now, after this finitistic digression, we return to predicativism.

Reverse mathematics for ACA_0.

ACA_0 is equivalent over RCA_0 to each of the following statements:

1. Every bounded, or bounded increasing, sequence of real numbers has a least upper bound.
2. The Bolzano/Weierstraß Theorem: Every bounded sequence of real numbers, or of points in $\mathbb{R}^d$, has a convergent subsequence.
3. Every sequence of points in a compact metric space has a convergent subsequence.
4. The Ascoli Lemma: Every bounded equicontinuous sequence of real-valued continuous functions on a bounded interval has a uniformly convergent subsequence.
5. Every countable commutative ring has a maximal ideal.
6. Every countable vector space over $\mathbb{Q}$, or over any countable field, has a basis.
7. Every countable field (of characteristic 0) has a transcendence basis.
8. Every countable Abelian group has a unique divisible closure.
9. König's Lemma for finitely branching trees.
10. Ramsey's Theorem for colorings of $[\mathbb{N}]^3$, or of $[\mathbb{N}]^4$, $[\mathbb{N}]^5$, ...

An open problem concerning ACA_0.

For any set $X \subseteq \mathbb{N}$, let $\mathsf{FS}(X) =$ the set of all sums of nonempty finite subsets of X. Hindman's Theorem says the following. Given a k-coloring $\mathbb{N} = C_1 \cup \cdots \cup C_k$, there exists an infinite set $X \subseteq \mathbb{N}$ such that $\mathsf{FS}(X) \subseteq C_i$ for some i.

Reverse-mathematically, it is known that Hindman's Theorem lies somewhere between ACA_0 and a slightly stronger system ACA_0^+ (BHS 1987). A fascinating and long-standing open problem is to learn whether Hindman's Theorem is provable in ACA_0.

There are at least four known proofs of Hindman's Theorem. So far as we know, the only proof that is formalizable in ACA_0^+ is the least elegant one. The most elegant one uses idempotent ultrafilters. Recently Montalbán and Shore did a reverse-mathematical study of the ultrafilter proof, but they did not answer the ACA_0 problem.

Reverse Mathematics for ATR_0.

ATR_0 is equivalent over RCA_0 to each of the following statements:

1. Any two countable well orderings are comparable.
2. Ulm's Theorem: Every countable reduced Abelian p-group is characterized up to isomorphism by its Ulm invariants.
3. The Perfect Set Theorem: Every uncountable closed, or analytic, set has a perfect subset.
4. Lusin's Separation Theorem: Any two disjoint analytic sets can be separated by a Borel set.
5. The domain of any single-valued Borel set in the plane is a Borel set.
6. Every clopen (or open) game in $\mathbb{N}^{\mathbb{N}}$ is determined.
7. Ramsey's Theorem for clopen (or closed) 2-colorings of $[\mathbb{N}]^{\infty}$.
8. Podewski-Steffens Theorem: Given a countable bipartite graph, there exist a matching M and a vertex covering C such that C consists of exactly one vertex from each edge of M.

A remark on predicative reductionism.

All of the above statements are impredicative, because they are not provable in IR. (E.g., they are false in the ω-model of IR consisting of the hyperarithmetical hierarchy up to Γ_0.) However, all of these statements are provable in ATR_0, and ATR_0 is Π^1_1-conservative over IR. Thus we have a partial realization of a program of predicative reductionism.

An open problem concerning ATR_0.

According to a 1948 conjecture of Fraïssé, the set of all countable linear orderings under embeddability form a *well-quasi-ordering*, i.e., there is no infinite descending chain and no infinite antichain. Fraïssé's Conjecture is now a theorem, proved by Richard Laver in 1971.

The reverse-mathematical status of Fraïssé's Conjecture is a fascinating and long-standing open problem. It is known that Fraïssé's Conjecture implies ATR_0. It is open whether Fraïssé's Conjecture is provable in ATR_0. Recently Montalbán made dramatic progress by showing that Fraïssé's Conjecture is strictly weaker than Π^1_1-CA_0.

Reverse Mathematics for Π^1_1-CA_0.

Π^1_1-CA_0 is equivalent over RCA_0 to each of the following statements:

1. Every tree has a largest perfect subtree.
2. The Cantor/Bendixson Theorem: Every closed subset of $\mathbb{R}$, or of any complete separable metric space, is the union of a countable set and a perfect set.
3. Every countable Abelian group is the direct sum of a divisible group and a reduced group.
4. Every difference of two open sets in the Baire space $\mathbb{N}^{\mathbb{N}}$ is determined.
5. Ramsey's Theorem for G_δ subsets of $[\mathbb{N}]^\infty$.

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