Recursion theory and symbolic dynamics

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Symbolic dynamics.

Let G be $(\mathbb{N}^d, +)$ or $(\mathbb{Z}^d, +)$ where $d \geq 1$.

Let A be a finite set of symbols.

We endow A with the discrete topology and A^G with the product topology.

The *shift action* of G on A^G is given by $(S^g x)(h) = x(g + h)$ for $g, h \in G$ and $x \in A^G$.

A subshift is a nonempty set $X \subseteq A^G$ which is topologically closed and *shift-invariant*, i.e., $x \in X$ implies $S^g x \in X$ for all $g \in G$.

Symbolic dynamics is the study of subshifts.

If $X \subseteq A^G$ and $Y \subseteq B^G$ are G-subshifts, a shift morphism from X to Y is a continuous mapping $\Phi : X \to Y$ such that $\Phi(S^g x) = S^g \Phi(x)$ for all $x \in X$ and $g \in G$.

By compactness, any shift morphism Φ is given by a *block code*, i.e., a finite mapping $\phi : A^F \to B$ where F is a finite subset of G and $\Phi(x)(g) = \phi(S^g x \restriction F)$ for all $x \in X$ and $g \in G$.

Some new (!?!) results on subshifts:

Let d be a positive integer, let A be a finite set of symbols, and let X be a nonempty subset of A^G where G is $\mathbb{N}^d$ or $\mathbb{Z}^d$.

The Hausdorff dimension, $\dim(X)$, and the effective Hausdorff dimension, $\mathrm{effdim}(X)$, are defined as usual with respect to the standard metric $\rho(x,y) = 2^{-|F_n|}$ where n is as large as possible such that $x{\restriction}F_n = y{\restriction}F_n$.

Here F_n is $\{1,\ldots,n\}^d$ if $G = \mathbb{N}^d$,
or $\{-n,\ldots,n\}^d$ if $G = \mathbb{Z}^d$.

We first state some old results.

1. $\mathrm{effdim}(X) = \sup_{x \in X} \mathrm{effdim}(x)$.
2. $\mathrm{effdim}(x) = \liminf_{n\to\infty} \dfrac{\mathsf{K}(x{\restriction}F_n)}{|F_n|}$.
3. $\mathrm{effdim}(X) = \dim(X)$
   provided X is *effectively closed*, i.e., Π^0_1.

Here K denotes Kolmogorov complexity.

Theorem 1 (Simpson 2010)**.** Assume that X is a <u>subshift</u>, i.e., closed and shift-invariant. Then

$$\mathsf{effdim}(X) = \dim(X) = \mathsf{ent}(X).$$

Moreover

$$\dim(X) \geq \limsup_{n\to\infty} \frac{\mathsf{K}(x{\upharpoonright}F_n)}{|F_n|} \quad \text{for all } x \in X,$$

and

$$\dim(X) = \lim_{n\to\infty} \frac{\mathsf{K}(x{\upharpoonright}F_n)}{|F_n|} \quad \text{for many } x \in X.$$

Remark. Here $\mathsf{ent}(X)$ denotes <u>entropy</u>,

$$\mathsf{ent}(X) = \lim_{n\to\infty} \frac{\log_2 |\{x{\upharpoonright}F_n \mid x \in X\}|}{|F_n|}.$$

This is known to be a conjugacy invariant.

Note. In the above theorem, there is no finiteness or computability hypothesis on the subshift X. Moreover, X can be a G-subshift where G is $\mathbb{N}^d$ or $\mathbb{Z}^d$ for any positive integer d.

Remark. The proof of Theorem 1 involves ergodic theory (Shannon/McMillan/Breiman, the Variational Principle, etc.) plus a combinatorial argument which is similar to the proof of the Vitali Covering Lemma.

Remark. Theorem 1 seems so fundamental that it could have been noticed long ago. Nevertheless, I have not been able to find it in the literature. So far as I can tell, everything in the theorem is new, except the following result of Furstenberg 1967:

$$\mathrm{dim}(X) = \mathrm{ent}(X) \text{ provided } G = \mathbb{N}.$$

The proof of this special case is much easier.

Remark. Theorem 1 is an outcome of my discussions at Penn State during February–April 2010 with many people including John Clemens, Mike Hochman, Dan Mauldin, Jan Reimann, and Sasha Shen.

Degrees of unsolvability (Muchnik).

Let X be any set of reals. We view X as a mass problem, viz., the problem of "finding" *some* $x \in X$.

In order to interpret "finding," we use Turing's concept of computability.

Accordingly, we say that X is algorithmically solvable if X contains *some* computable real, or in other words, $X \cap \mathrm{REC} \neq \emptyset$.

Similarly, X is algorithmically reducible to Y if *each* $y \in Y$ can be used as a Turing oracle to compute *some* $x \in X$.

The degree of unsolvability of X, $\deg(X)$, is the equivalence class of X under mutual algorithmic reducibility.

Reference:

Albert A. Muchnik, On strong and weak reducibilities of algorithmic problems, Sibirskii Matematicheskii Zhurnal, 4, 1963, 1328–1341, in Russian.

I propose applying recursion-theoretic concepts such as Muchnik degrees and Kolmogorov complexity to the study of symbolic dynamics.

Muchnik degrees of subshifts.

A subshift X is <u>of finite type</u> if it is given by a finite set of excluded finite configurations:

$$X = \{x \in A^G \mid (\forall g \in G)\,(S^g x{\restriction}F \notin E)\}$$

where E and F are finite.

Let $\mathcal{E}_{\mathsf{w}}$ be the lattice of Muchnik degrees of nonempty, effectively closed sets of reals.

$\mathcal{E}_{\mathsf{w}}$ is known to include many specific, natural degrees which are associated with foundationally interesting topics.

A picture of $\mathcal{E}_{\mathsf{w}}$ is on slides 10, 12, . . . , 20.

Theorem 2 (Simpson 2007)**.** The Muchnik degrees in $\mathcal{E}_{\mathsf{w}}$ are precisely the Muchnik degrees of $\mathbb{Z}^2$-subshifts of finite type.

Proof. One direction is trivial, because subshifts of finite type may be viewed as effectively closed sets. My proof of the other direction uses tiling techniques which go back to Berger 1966, Robinson 1971, Myers 1974. Another proof, Durand/Romashchenko/Shen 2008, uses "self-replicating tile sets."

Corollary (Simpson 2007)**.** We can construct an infinite family of $\mathbb{Z}^2$-subshifts of finite type which are strongly independent with respect to shift morphisms, etc.

Proof. This follows from the existence of an infinite independent set of degrees in $\mathcal{E}_{\mathrm{w}}$. The existence of such degrees was originally proved by means of a priority argument.

Thus we have an application of recursion theory (tiling methods plus priority argument) to prove a result in symbolic dynamics which does not mention computability concepts.

A possibly interesting research program:

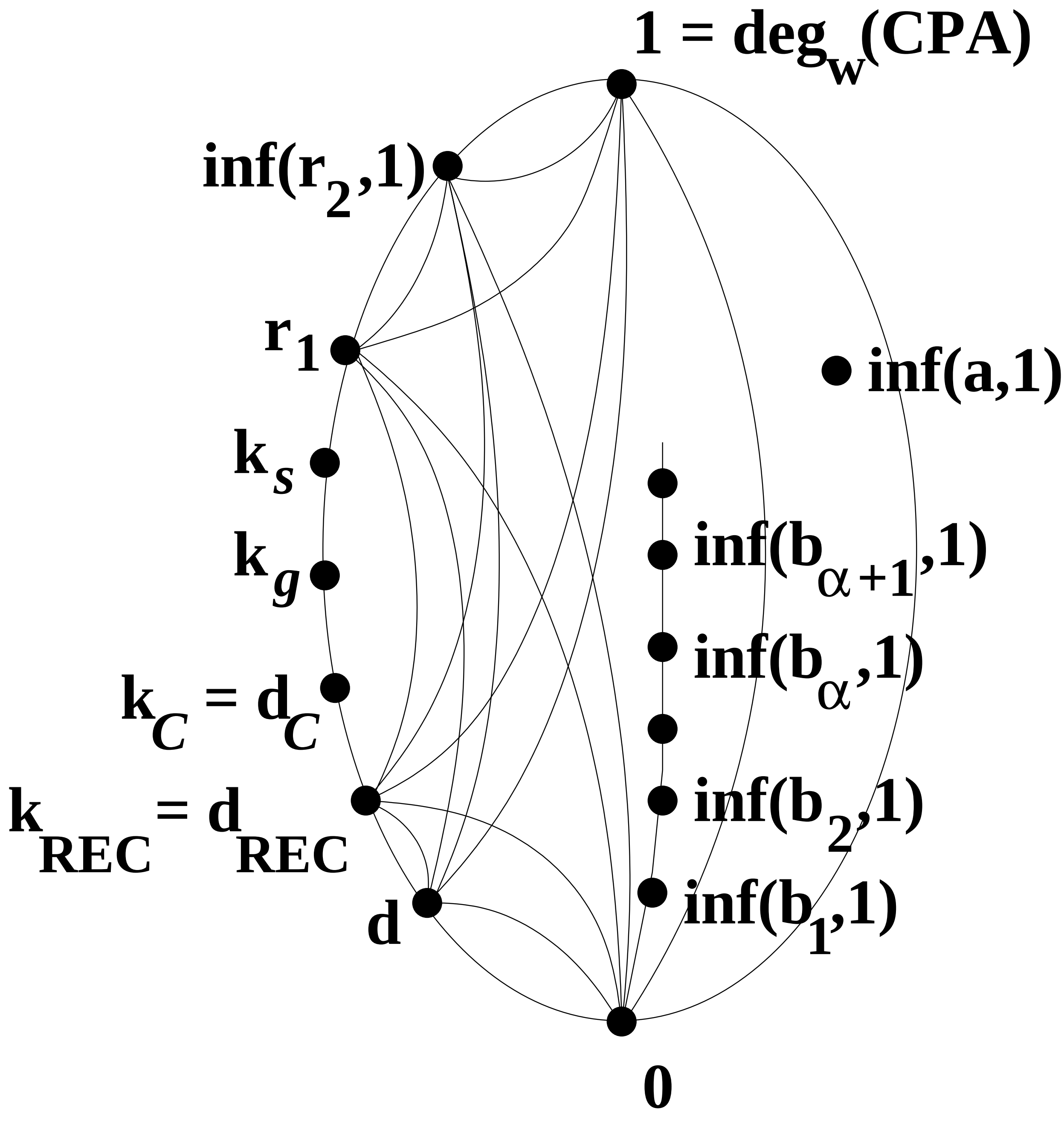
Given a subshift X, explore the relationship between the dynamical properties of X and the degree of unsolvability of X, i.e., its Muchnik degree, deg(X).

For example, the *entropy* of X is a well-known dynamical property which serves as an upper bound on the complexity of orbits. In particular ent(X) > 0 implies $(\exists x \in X)\,(x$ is not computable).

By contrast, the degree of unsolvability of X serves as a lower bound on the complexity of orbits. For instance, deg(X) $> \mathbf{0} \iff$ $(\forall x \in X)\,(x$ is not computable).

Theorem (Hochman)**.** If X is of finite type and *minimal* (i.e., every orbit is dense), then deg(X) $= \mathbf{0}$.

Actually this holds for all effectively closed subshifts, not necessarily of finite type.



A picture of $\mathcal{E}_{\mathsf{w}}$. Each black dot except $\inf(\mathbf{a},\mathbf{1})$ represents a specific, natural degree in $\mathcal{E}_{\mathsf{w}}$. We shall explain some of these degrees.

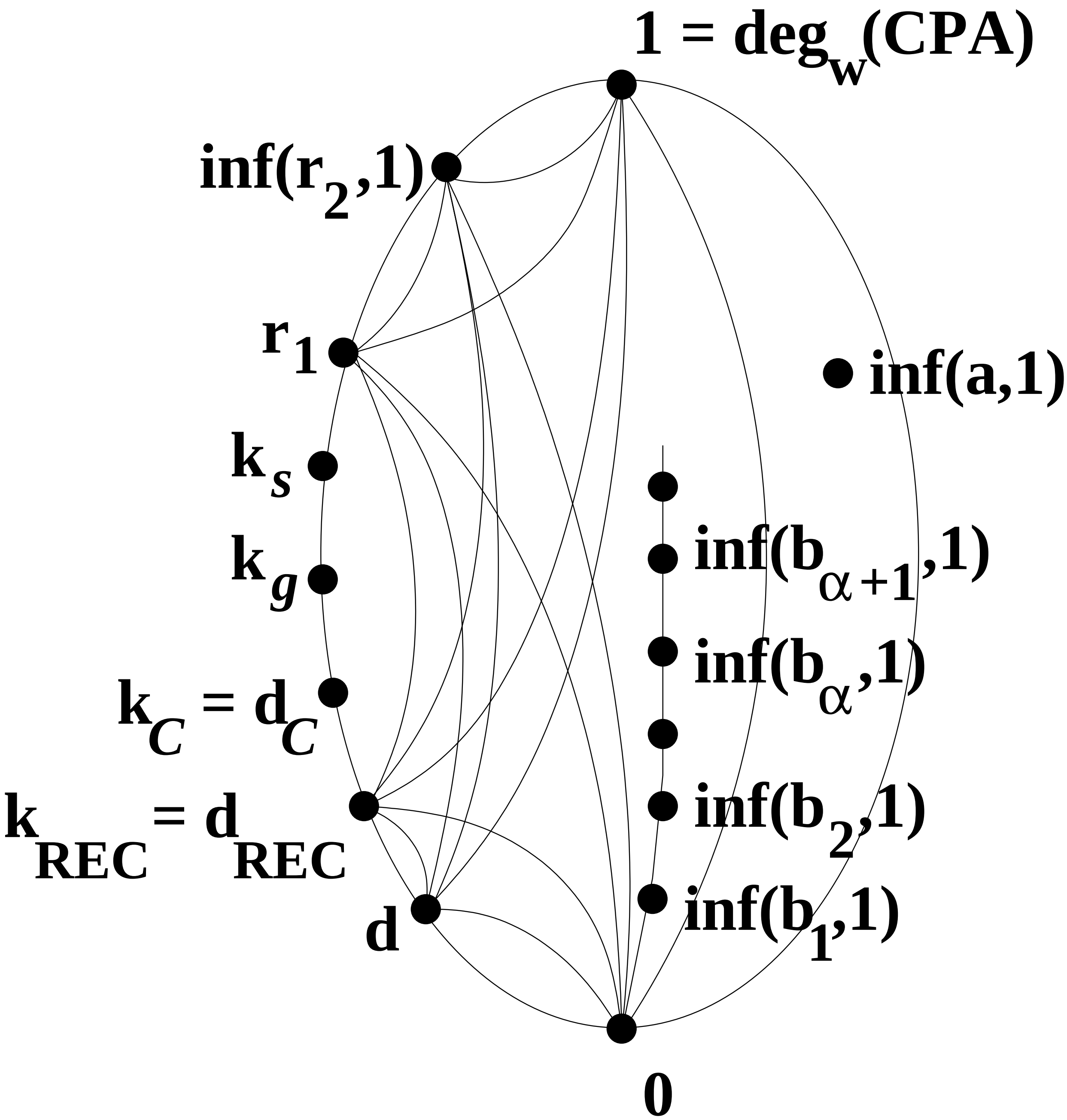
Two subshifts are said to be <u>conjugate</u> if they are topologically isomorphic, i.e., there is a shift isomorphism between them.

The basic problem of symbolic dynamics is: classify subshifts up to conjugacy invariance.

Muchnik degrees can help, because the Muchnik degree of a subshift is a <u>conjugacy invariant</u>. In particular, each degree in $\mathcal{E}_{\mathsf{w}}$ including $\mathbf{0}$, $\mathbf{1}$, $\mathbf{r}_1$, $\mathbf{d}$, $\mathbf{d}_{\mathsf{REC}}$, $\mathbf{d}_C$, $\mathbf{k}_s$, $\mathbf{k}_g$, $\inf(\mathbf{r}_2, \mathbf{1})$, $\inf(\mathbf{b}_\alpha, \mathbf{1})$, and even $\inf(\mathbf{a}, \mathbf{1})$ may be viewed as a conjugacy invariant for subshifts of finite type.

It is interesting to compare the Muchnik degree of a subshift X with other conjugacy invariants, e.g., the entropy of X.

Generally speaking, the Muchnik degree of X represents a lower bound on the complexity of the orbits, while the entropy of X is an upper bound on the complexity of these same orbits.



A picture of $\mathcal{E}_{\mathrm{w}}$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{r} =$ randomness, $\mathbf{b} =$ LR-reducibility, $\mathbf{k} =$ complexity, $\mathbf{d} =$ diagonal nonrecursiveness.

We now explain some degrees in $\mathcal{E}_\mathsf{w}$.

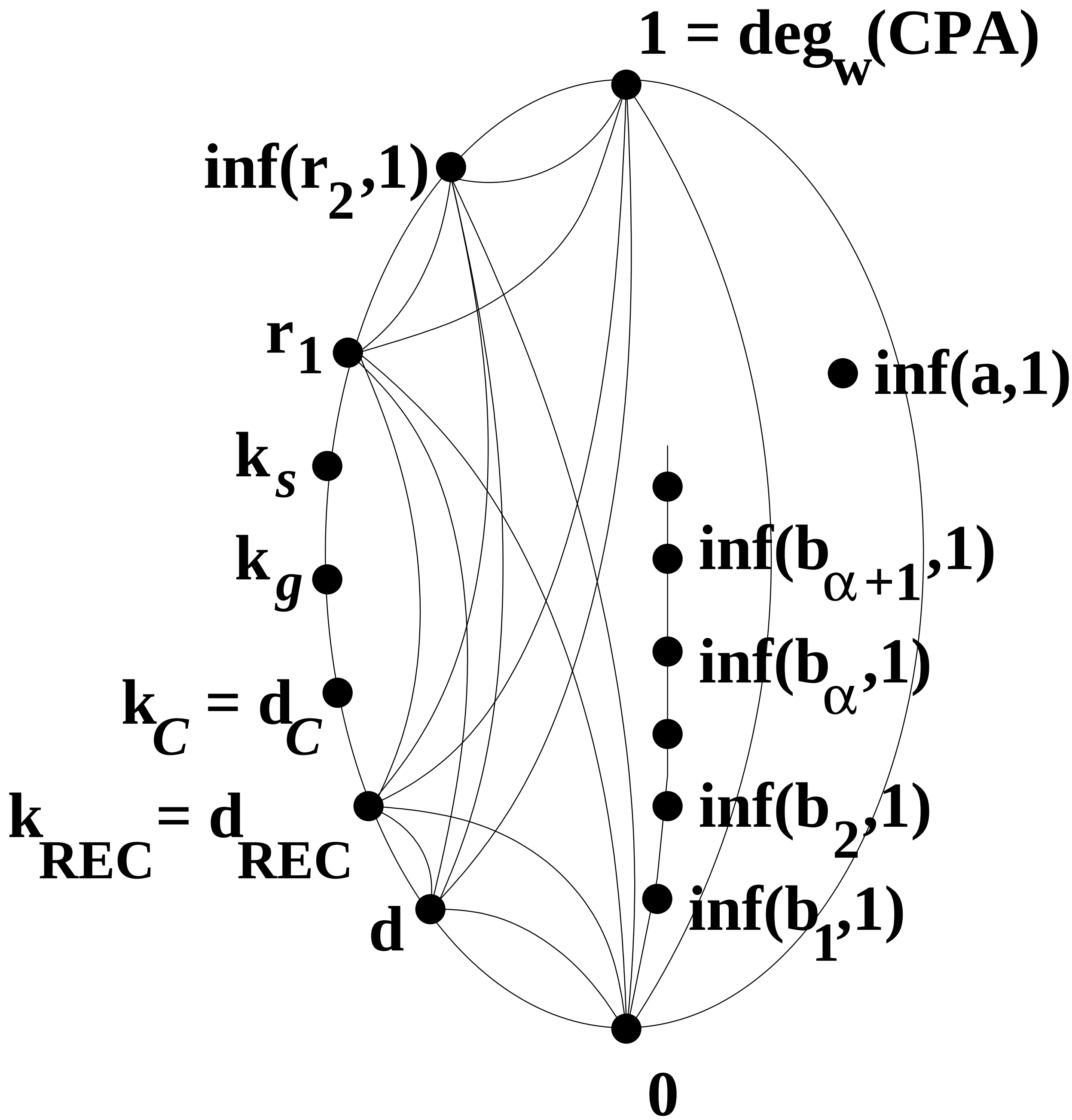
The top degree in $\mathcal{E}_\mathsf{w}$ is $\mathbf{1} =$ deg(CPA) where CPA is the problem of finding a complete consistent theory which includes Peano Arithmetic (or ZFC, etc.).

We also have inf$(\mathbf{a}, \mathbf{1}) \in \mathcal{E}_\mathsf{w}$ where $\mathbf{a}$ is any recursively enumerable Turing degree. Moreover, $\mathbf{a} < \mathbf{b}$ implies inf$(\mathbf{a}, \mathbf{1}) <$ inf$(\mathbf{b}, \mathbf{1})$.

We have $\mathbf{r}_1 \in \mathcal{E}_\mathsf{w}$ where $\mathbf{r}_1 =$ deg(MLR), MLR $= \{x \in \{0,1\}^\mathbb{N} \mid x$ is <u>random</u>$\})$, i.e., random in the sense of Martin-Löf.

We also have inf$(\mathbf{r}_2, \mathbf{1}) \in \mathcal{E}_\mathsf{w}$ where $\mathbf{r}_2 =$ deg$(\{x \in \{0,1\}^\mathbb{N} \mid x$ is <u>2-random</u>$\})$, i.e., random relative to the halting problem.

Also $\mathbf{d} \in \mathcal{E}_\mathsf{w}$ where $\mathbf{d} =$ deg$(\{f \in \mathbb{N}^\mathbb{N} \mid f$ is <u>diagonally nonrecursive</u>$\})$, i.e., $\forall n\,(f(n) \neq \varphi_n(n))$.



A picture of $\mathcal{E}_\mathrm{w}$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{r} =$ randomness, $\mathbf{b} =$ LR-reducibility, $\mathbf{k} =$ complexity, $\mathbf{d} =$ diagonal nonrecursiveness.

Let REC $= \{g \in \mathbb{N}^{\mathbb{N}} \mid g$ is recursive$\}$.
Let C be any "nice" subclass of REC.
For instance $C =$ REC, or $C = \{g \in$ REC $\mid$ g is primitive recursive$\}$. We have $\mathbf{d}_C \in \mathcal{E}_{\mathsf{w}}$ where $\mathbf{d}_C = \deg(\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is diagonally nonrecursive and <u>C-bounded</u>$\})$,
i.e., $(\exists g \in C)\, \forall n\, (f(n) < g(n))$.

Also, $\mathbf{d}_C = \deg(\{x \in \{0,1\}^{\mathbb{N}} \mid x$ is <u>C-complex</u>$\}$,
i.e., $(\exists g \in C)\, \forall n\, (\mathsf{K}(x{\restriction}\{1,\ldots,g(n)\}) \geq n)\})$.
Moreover, $\mathbf{d}_{C'} < \mathbf{d}_C$ whenever C' contains a function which dominates all functions in C.

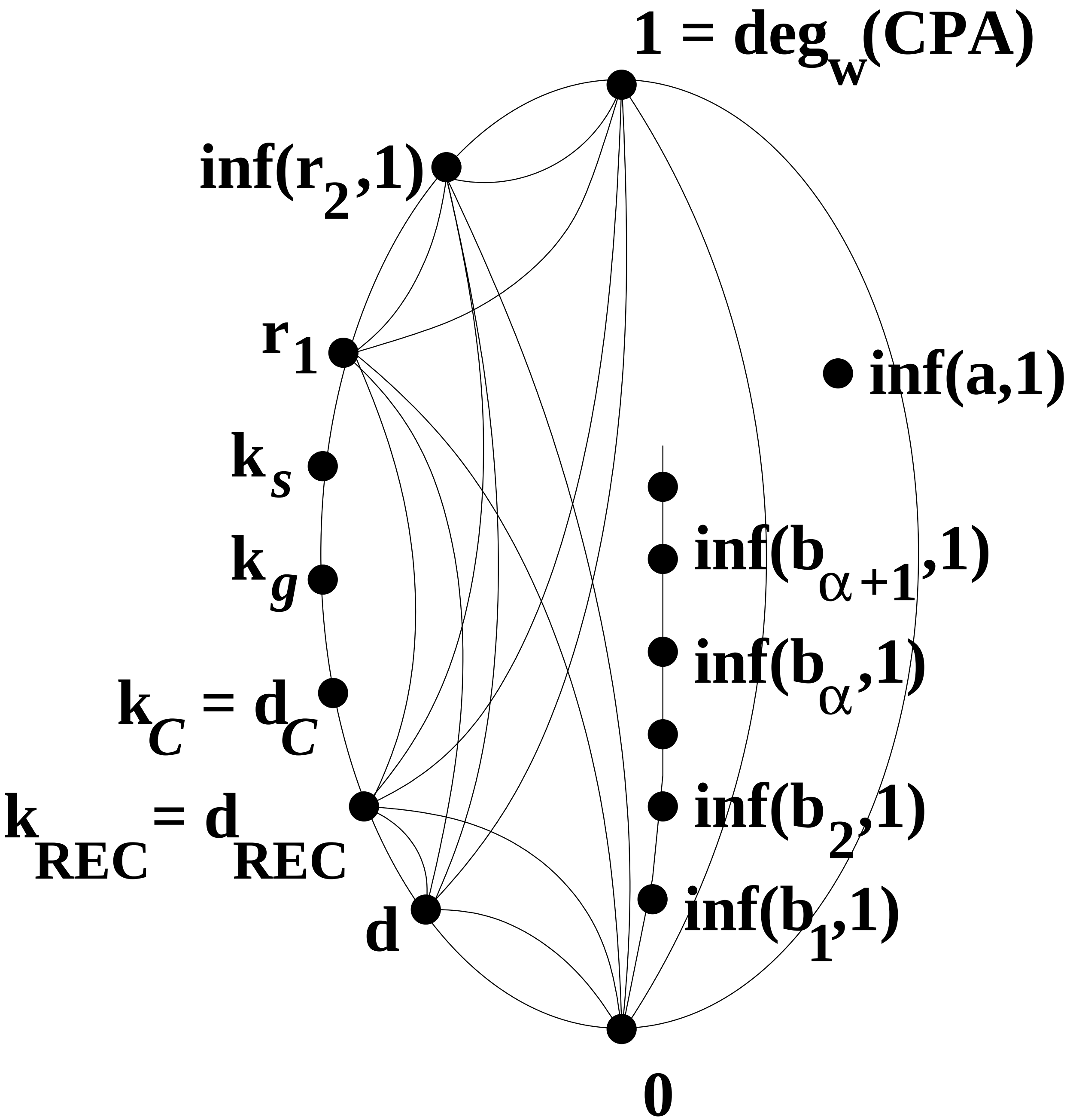
For $x \in \{0,1\}^{\mathbb{N}}$ let effdim$(x) =$ the <u>effective Hausdorff dimension</u> of x, i.e.,

$$\mathrm{effdim}(x) = \liminf_{n\to\infty} \frac{\mathsf{K}(x{\restriction}\{1,\ldots,n\})}{n}.$$

Given a right recursively enumerable real number $s < 1$, we have $\mathbf{k}_s \in \mathcal{E}_{\mathsf{w}}$ where

$$\mathbf{k}_s = \deg(\{x \in \{0,1\}^{\mathbb{N}} \mid \mathrm{effdim}(x) > s\}).$$

Moreover, $s < t$ implies $\mathbf{k}_s < \mathbf{k}_t$ (Miller).



A picture of $\mathcal{E}_\mathrm{w}$. Here $\mathbf{a}$ = any r.e. degree, $\mathbf{r}$ = randomness, $\mathbf{b}$ = LR-reducibility, $\mathbf{k}$ = complexity, $\mathbf{d}$ = diagonal nonrecursiveness.

More generally, let $g : \mathbb{N} \to [0, \infty)$ be an unbounded computable function such that $g(n) \le g(n+1) \le g(n)+1$ for all n.

For example, $g(n)$ could be $n/2$ or $n/3$ or $\sqrt{n}$ or $\sqrt[3]{n}$ or $\log n$ or $\log n + \log\log n$ or $\log\log n$ or the inverse Ackermann function.

Define $\mathbf{k}_g = \deg(\{x \mid x \text{ is } \underline{g\text{-random}}\})$, i.e., $\exists c \, \forall n \, (\mathsf{K}(x{\upharpoonright}\{1, \ldots, n\} \ge g(n) - c)$.

Theorem (Hudelson 2010). $\mathbf{k}_g < \mathbf{k}_h$ provided $g(n) + 2\log_2 g(n) \le h(n)$ for all n.

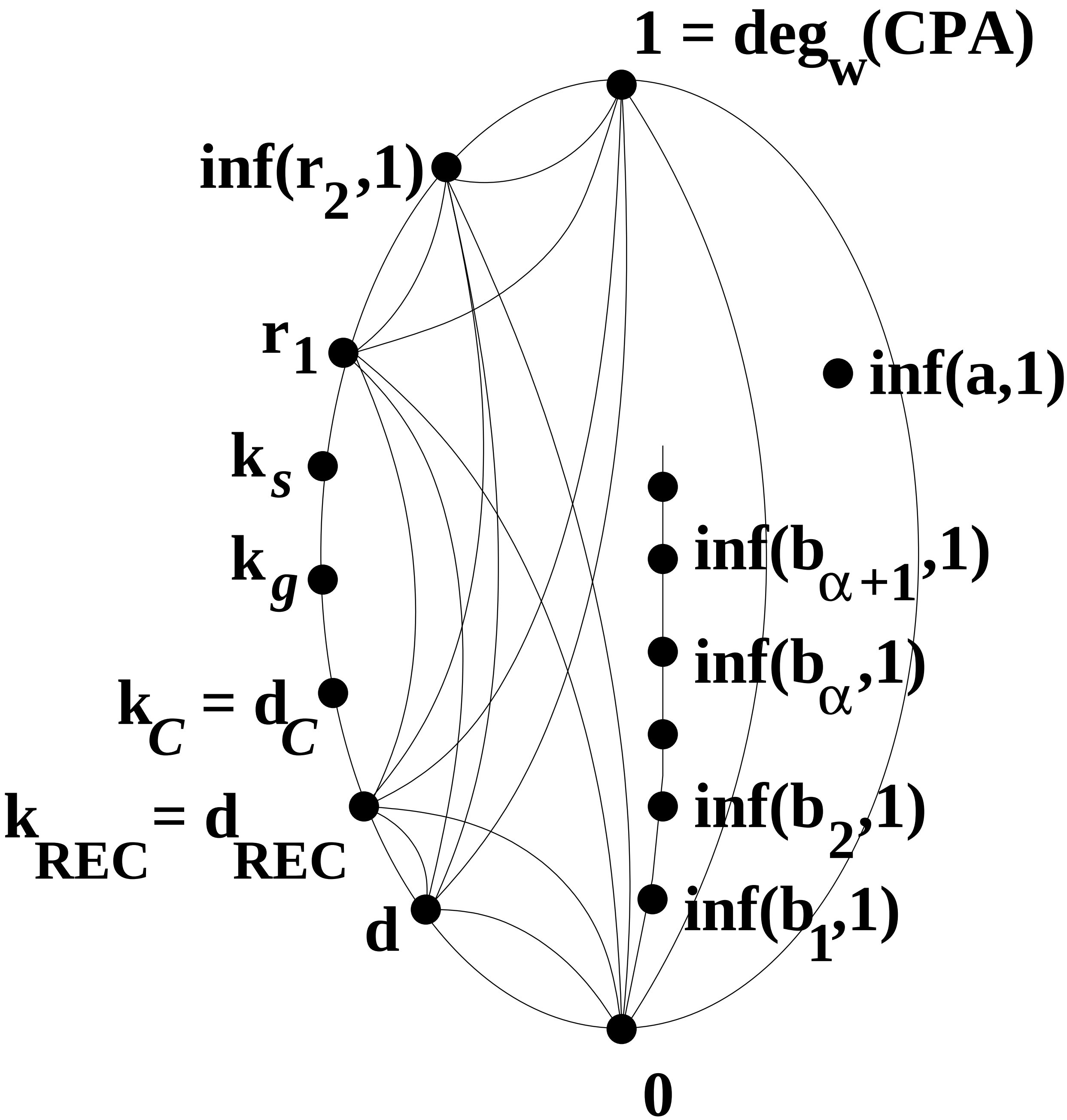
In other words, there exists a g-random real with no h-random real Turing reducible to it.

This is a generalization of Miller's theorem on the difficulty of information extraction.

References:

Phil Hudelson, Mass problems and initial segment complexity, Journal of Symbolic Logic, to appear

Joseph S. Miller, Extracting information is hard, Advances in Mathematics, 226, 2011, 373–384.



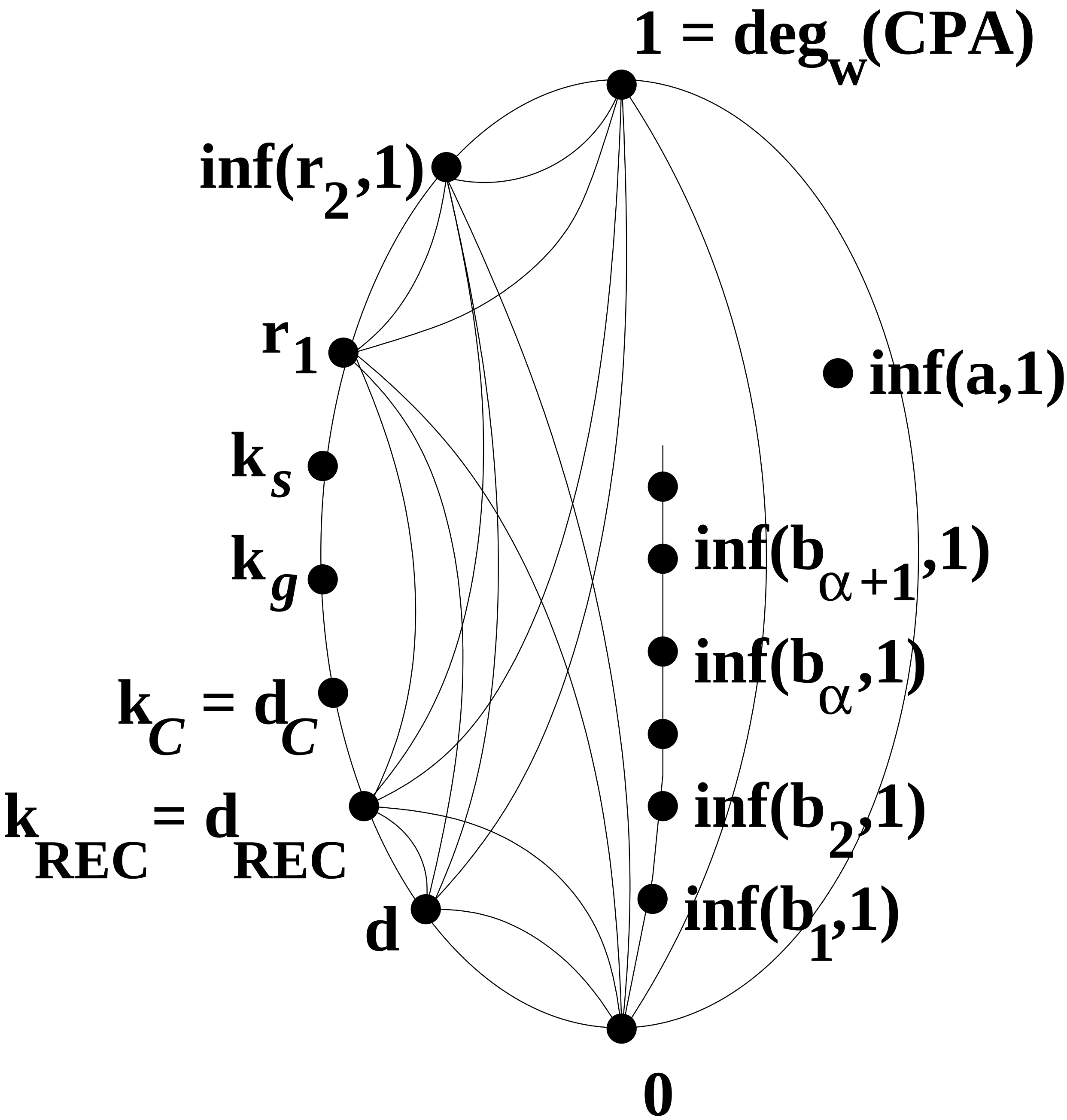
A picture of $\mathcal{E}_{\mathrm{w}}$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{r} =$ randomness, $\mathbf{b} =$ LR-reducibility, $\mathbf{k} =$ complexity, $\mathbf{d} =$ diagonal nonrecursiveness.

Letting z be a Turing oracle, define
$\mathsf{MLR}^z = \{x \mid x \text{ is random relative to } z\}$
and $\mathsf{K}^z(\tau) =$ the prefix-free Kolmogorov complexity of τ relative to z.
Define $y \leq_{\mathsf{LR}} z \iff \mathsf{MLR}^z \subseteq \mathsf{MLR}^y$
and $y \leq_{\mathsf{LK}} z \iff \exists c \, \forall \tau \, (\mathsf{K}^z(\tau) \leq \mathsf{K}^y(\tau) + c)$.

Theorem (Miller/Kjos-Hanssen/Solomon)**.**
We have $y \leq_{\mathsf{LR}} z \iff y \leq_{\mathsf{LK}} z$.

For each recursive ordinal number α, let
$0^{(\alpha)} =$ the αth iterated Turing jump of 0.
Thus $0^{(1)} =$ the halting problem, and
$0^{(\alpha+1)} =$ the halting problem relative to $0^{(\alpha)}$, etc. This is the <u>hyperarithmetical hierarchy</u>.
We embed it naturally into $\mathcal{E}_{\mathsf{W}}$ as follows.

Theorem (Simpson 2009)**.** $0^{(\alpha)} \leq_{\mathsf{LR}} z$
$\iff$ every $\Sigma^0_{\alpha+2}$ set includes a $\Sigma^{0,z}_2$ set of the same measure. Moreover,
letting $\mathbf{b}_\alpha = \deg(\{z \mid 0^{(\alpha)} \leq_{\mathsf{LR}} z\})$ we have
$\inf(\mathbf{b}_\alpha, \mathbf{1}) \in \mathcal{E}_{\mathsf{W}}$ and $\inf(\mathbf{b}_\alpha, \mathbf{1}) < \inf(\mathbf{b}_{\alpha+1}, \mathbf{1})$.



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History: Kolmogorov 1932 developed his "calculus of problems" as a nonrigorous yet compelling explanation of Brouwer's intuitionism. Later Medvedev 1955 and Muchnik 1963 proposed Medvedev degrees and Muchnik degrees as rigorous versions of Kolmogorov's idea.

Some references:

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THE END. THANK YOU!