On the Muchnik degrees of 2-dimensional subshifts of finite type

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Abstract.

We apply some concepts and results from mathematical logic in order to obtain an apparently new counterexample in symbolic dynamics.

Two sets are said to be *Muchnik equivalent* if any point of either set can be used as a Turing oracle to compute a point of the other set. The *Muchnik degree* of a set is its Muchnik equivalence class. There is an extensive recursion-theoretic literature on the lattice $\mathcal{P}_w$ of Muchnik degrees of nonempty, recursively closed subsets of the Cantor space. It is known that $\mathcal{P}_w$ contains many specific, interesting, Muchnik degrees related to various topics in the foundations of mathematics and the foundations of computer science. Moreover, the lattice-theoretical structure of $\mathcal{P}_w$ is fairly well understood.

We prove that $\mathcal{P}_w$ consists precisely of the Muchnik degrees of 2-dimensional subshifts of finite type. We use this result to obtain an infinite collection of 2-dimensional subshifts of finite type which are, in a certain sense, mutually incompatible.

We begin with the 1-dimensional case.

A *dynamical system* consists of a nonempty set X (the set of *states*) plus a mapping $T : X \to X$ (the *state transition operator*).

Throughout this talk we assume that X is compact and metrizable. We also assume that T is continuous, one-to-one, and onto.

Example. Let A be a finite set of symbols. $A^{\mathbb{Z}}$ is the set of bi-infinite sequences of symbols from A. The *shift operator* $S : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is given by $S(x)(n) = x(n+1)$ for all $x \in A^{\mathbb{Z}}$.

The dynamical system consisting of the compact metrizable space $A^{\mathbb{Z}}$ and the shift operator S is known as the *full shift* on A.

Let X be a nonempty closed subset of $A^{\mathbb{Z}}$ which is *invariant under the shift operator*, i.e., $x \in X \Longleftrightarrow S(x) \in X$ for all x.

The dynamical system consisting of the compact metrizable space X together with the shift operator S (actually $S \upharpoonright X$)
is known as a *subshift* on A.
It is a subsystem of the full shift on A.

There are many different kinds of subshifts.
Subshifts are very useful for describing
the behavior of dynamical systems in general.

The study of subshifts for their own sake is called *symbolic dynamics*.

References:

Mike Boyle, Algebraic aspects of symbolic dynamics, in *Topics in Symbolic Dynamics and Applications*, London Mathematical Society Lecture Notes Series, 2000, pages 57–88.

Douglas Lind and Brian Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, 1995, XVI + 495 pages.

Every subshift $X \subseteq A^{\mathbb{Z}}$ is defined by a set of *excluded words*. Namely, for an appropriate set E of finite sequences of symbols from A,

$$X = \{x \in A^{\mathbb{Z}} \mid x \text{ contains no consecutive subsequence belonging to } E\}.$$

If E is finite, we say that X is *of finite type*.

Subshifts of finite type have been studied extensively. It is easy to see that every 1-dimensional subshift of finite type contains periodic points.

If E is *recursive* (computable), then X is Π^0_1.

Cenzer/Dashti/King 2007 have constructed a 1-dimensional Π^0_1 subshift which contains no recursive points, hence no periodic points.

Many questions regarding 1-dimensional Π^0_1 subshifts remain open.

We now turn to the 2-dimensional case.

As before, let A be a finite set of symbols.

Let $A^{\mathbb{Z}\times\mathbb{Z}}$ be the set of doubly bi-infinite double sequences of symbols from A.

This is again a compact metrizable space.

Points of $A^{\mathbb{Z}\times\mathbb{Z}}$ may be viewed as *tilings of the plane*, in the sense of Wang 1961.

Tiling problems were studied by logicians during the years 1960–1980.

The connection with dynamical systems was noticed only relatively recently.

A *2-dimensional dynamical system* consists of a nonempty set X and a commuting pair of maps $T_1, T_2 : X \to X$. As before we assume X compact metrizable, T_1, T_2 continuous one-to-one onto.

The *full 2-dimensional shift* on A is the dynamical system consisting of $A^{\mathbb{Z}\times\mathbb{Z}}$ with shift operators $S_1, S_2 : A^{\mathbb{Z}\times\mathbb{Z}} \to A^{\mathbb{Z}\times\mathbb{Z}}$ given by
$S_1(x)(m,n) = x(m+1,n)$ and
$S_2(x)(m,n) = x(m,n+1)$.

A *2-dimensional subshift* on A is a nonempty closed set $X \subseteq A^{\mathbb{Z}\times\mathbb{Z}}$ which is invariant under S_1 and S_2.

Note that (X, S_1, S_2) is again a 2-dimensional dynamical system. It is a subsystem of the full 2-dimensional shift on A.

As in the 1-dimensional case, every 2-dimensional subshift X is defined by a set E of excluded configurations.

If E is finite, X is said to be *of finite type*.

Here, by a *configuration* we mean a "2-dimensional word," i.e., a member of $A^{\{1,\ldots,r\}\times\{1,\ldots,r\}}$ for some positive integer r.

2-dimensional subshifts of finite type are important in dynamical systems theory. An example is the Ising model in mathematical physics.

History:

Berger 1966 answered a question of Wang 1961 by constructing a 2-dimensional subshift of finite type with no periodic points.

Berger 1966 showed that it is undecidable whether a given finite set of excluded configurations defines a (nonempty!) 2-dimensional subshift.

Myers 1974 constructed
a 2-dimensional subshift of finite type
with no recursive points.

Hochman/Meyerovitch 2007 proved:
a real number $h \geq 0$ is the entropy of
a 2-dimensional subshift of finite type if and only if h is *right recursively enumerable*.
This means that h is the limit of a recursive decreasing sequence of rational numbers.

Using the methods of Robinson 1971 and Myers 1974, I have proved:

Theorem 1 (Simpson 2007)**.** The Muchnik degrees of 2-dimensional subshifts of finite type are the same as the Muchnik degrees of nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$.

This theorem is useful, because we can then apply known results from recursion theory to study 2-dimensional subshifts of finite type.

Below we shall present one such application.

Our application will be stated purely in terms of 2-dimensional subshifts of finite type, with no mention of Muchnik degrees and no mention of recursion theory.

References:

Hao Wang, Proving theorems by pattern recognition, II, *Bell System Technical Journal*, 40, 1–42, 1961.

Robert Berger, The undecidability of the domino problem, *Memoirs of the American Mathematical Society*, 66, 72 pages, 1966.

Raphael M. Robinson, Undecidability and nonperiodicity of tilings of the plane, *Inventiones Mathematicae*, 12, 177–209, 1971.

Dale Myers, Nonrecursive tilings of the plane, II. *Journal of Symbolic Logic*, 39, 286–294, 1974.

Michael Hochman and Tom Meyerovitch, A characterization of the entropies of multidimensional shifts of finite type, arXiv:math:DS/0703206v1, 27 pages, 2007.

Douglas Cenzer, Ali Dashti, and Jonathan King, Effective symbolic dynamics, 12 pages, 2007.

Stephen G. Simpson, Medvedev degrees of 2-dimensional subshifts of finite type, preprint, 8 pages, 1 May 2007, *Ergodic Theory and Dynamical Systems*, to appear.

To state our application, we need some easy definitions which make perfect sense for all dynamical systems.

Definition. Let X and Y be 2-dimensional subshifts on k and l symbols respectively. The Cartesian product $X \times Y$ and the disjoint union $X + Y$ are 2-dimensional subshifts on kl and $k + l$ symbols respectively.

Definition. Let (X, S_1, S_2) be a 2-dimensional subshift on k symbols. Let a, b, c, d be integers with $ad - bc \neq 0$. Then, the system $(X, S_1^a S_2^b, S_1^c S_2^d)$ is canonically isomorphic to a 2-dimensional subshift on $k^{|ad-bc|}$ symbols.

Definition. If $\mathcal{U}$ is a set of 2-dimensional subshifts, let cl($\mathcal{U}$) be the closure of $\mathcal{U}$ under the above operations.

Definition. If X and Y are 2-dimensional subshifts, a *shift morphism* from X to Y is a continuous mapping $F : X \to Y$ which commutes with the shift operators.

In other words, $F(S_1(x)) = S_1(F(x))$ and $F(S_2(x)) = S_2(F(x))$ for all $x \in X$.

Now for the application.

Theorem 2 (Simpson 2007)**.**
There is an infinite set $\mathcal{W}$ of 2-dimensional subshifts of finite type, such that for any partition $\mathcal{U}, \mathcal{V}$ of $\mathcal{W}$, and for any $X \in \text{cl}(\mathcal{U})$ and $Y \in \text{cl}(\mathcal{V})$, there is no shift morphism from X to Y or vice versa.

Theorem 2 follows from Theorem 1 plus a previously known recursion-theoretic result:

> There is an infinite set of nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ whose Muchnik degrees are independent.

This known recursion-theoretic result is proved by means of a priority argument.

We shall now discuss some ingredients of Theorems 1 and 2 and their proofs.

The essential concept from recursion theory is as follows.

A *recursive functional* is a mapping F from a subset of $\{0,1\}^{\mathbb{N}}$ to a subset of $\{0,1\}^{\mathbb{N}}$ which is defined by a finite, deterministic, computer program $\mathcal{P}$ in the following way:

For all points $x, \in \{0,1\}^{\mathbb{N}}$ and $y \in \{0,1\}^{\mathbb{N}}$, $F(x) = y$ if and only if for each $n \in \mathbb{N}$ the run of the program $\mathcal{P}$ with input n using x as a Turing oracle eventually halts with output $y(n)$.

Instead of the Cantor space $\{0,1\}^{\mathbb{N}}$ we may use any of the spaces $A^{\mathbb{Z}}$ or $A^{\mathbb{Z}\times\mathbb{Z}}$ where A is a finite set of symbols.

Let X and Y be subsets of
any of the spaces $\{0,1\}^{\mathbb{N}}$ or $A^{\mathbb{Z}}$ or $A^{\mathbb{Z}\times\mathbb{Z}}$.

X and Y are *recursively homeomorphic* if there exists a recursive functional F with a recursive inverse F^{-1} such that $X \subseteq \text{dom}(F)$ and $Y \subseteq \text{rng}(F)$ and F maps X one-to-one onto Y.

Y is *Muchnik reducible to* X if for each $x \in X$ there exists a recursive functional F such that $x \in \text{dom}(F)$ and $F(x) \in Y$.

X and Y are *Muchnik equivalent* if each is Muchnik reducible to the other.

Clearly recursive homeomorphism implies Muchnik equivalence, but the converse does not hold.

A *Muchnik degree* is an equivalence class under Muchnik equivalence.

A subset of $A^{\mathbb{Z}\times\mathbb{Z}}$ or of $A^{\mathbb{Z}}$ or of $\{0,1\}^{\mathbb{N}}$ is *recursively closed* if it is the complement of the union of a recursive sequence of basic open sets.

Here a *basic open set* is any set of the form $N_\sigma = \{x \mid x \restriction \mathrm{dom}(\sigma) = \sigma\}$ where σ is a finite function.

By definition, a set is Π^0_1 if and only if it is recursively closed.

Clearly the spaces $A^{\mathbb{Z}\times\mathbb{Z}}$ and $A^{\mathbb{Z}}$ and $\{0,1\}^{\mathbb{N}}$ are recursively homeomorphic to each other.

Hence, Π^0_1 sets in any of them are recursively homeomorphic to Π^0_1 sets in all of them.

Clearly subshifts of finite type are Π^0_1. More generally, any subshift defined by a recursive sequence of excluded configurations is Π^0_1.

The Muchnik degrees of all nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ under Muchnik reducibility form a countable distributive lattice, $\mathcal{P}_w$.

It is known that $\mathcal{P}_w$ is structurally rich.

For instance, every nonzero degree in $\mathcal{P}_w$ is join-reducible in $\mathcal{P}_w$ (Binns 2003).

Moreover, every countable distributive lattice is lattice-embeddable in every nontrivial initial segment of $\mathcal{P}_w$ (Binns/Simpson 2004).

Theorem 1 says that the Muchnik degrees of 2-dimensional subshifts of finite type are precisely the Muchnik degrees in $\mathcal{P}_w$.

By contrast, all 1-dimensional subshifts of finite type are of Muchnik degree zero.

Thus, the 2-dimensional case is much more complicated than the 1-dimensional case.

Let X and Y be 2-dimensional subshifts.

A basic fact concerning shift morphisms:

Each shift morphism $F : X \to Y$ is describable in a very simple manner as a *block code*. This means that $F(x)(m,n)$ depends only on $x(m \pm i, n \pm j)$, $i,j \in \{0,\ldots,r\}$ for some fixed r.

In particular, each shift morphism is given by a recursive functional. Thus, the existence of a shift morphism from X to Y implies that Y is Muchnik reducible to X.

Define $X \geq Y$ if there exists a shift morphism from X to Y. Define $X \equiv Y$ if $X \geq Y$ and $Y \geq X$. The $\equiv$-equivalence classes form a distributive lattice. We have:

Theorem 3 (Simpson 2007)**.** There is a canonical lattice homomorphism of the lattice of $\equiv$-equivalence classes of 2-dimensional subshifts of finite type, onto the lattice $\mathcal{P}_w$.

In all of these lattices, the supremum and infimum are given by $X \times Y$ and $X + Y$.

If X is a 2-dimensional subshift of finite type, there are surely some interesting relationships between the dynamical properties of X and the Muchnik degree of X.

These relationships remain to be explored.

Moreover, $\mathcal{P}_w$ contains a number of specific, natural, weak degrees which are linked to various interesting topics in the foundations of mathematics and the foundations of computer science.

- algorithmic randomness
- reverse mathematics
- almost everywhere domination
- diagonal nonrecursiveness
- hyperarithmeticity
- resource-bounded computational complexity
- Kolmogorov complexity
- effective Hausdorff dimension
- subrecursive hierarchies

Some examples of Muchnik degrees in $\mathcal{P}_w$:

$\mathbf{0}$ = the bottom degree in $\mathcal{P}_w$
= the Muchnik degree of $\{x \mid x$ is recursive$\}$

$\mathbf{1}$ = the top degree in $\mathcal{P}_w$ = the Muchnik degree of $\{x \mid x$ is a completion of Peano Arithmetic$\}$

$\mathbf{r}_1$ = Muchnik degree of $\{x \in \{0,1\}^{\mathbb{N}} \mid x$ is *random*$\}$ (in the sense of P. Martin-Löf)

$\mathbf{r}_2$ = Muchnik degree of $\{x \in \{0,1\}^{\mathbb{N}} \mid x$ is random relative to $0'$, the Halting Problem$\}$

$\mathbf{d}$ = the Muchnik degree of $\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is *diagonally nonrecursive*$\}$ (i.e., $f(n) \not\simeq \varphi_n^{(1)}(n)$ for all n)

$\mathbf{d}_{\mathsf{REC}}$ = the Muchnik degree of $\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is diagonally nonrecursive and *recursively bounded*$\}$ (i.e., f is bounded by a recursive function)

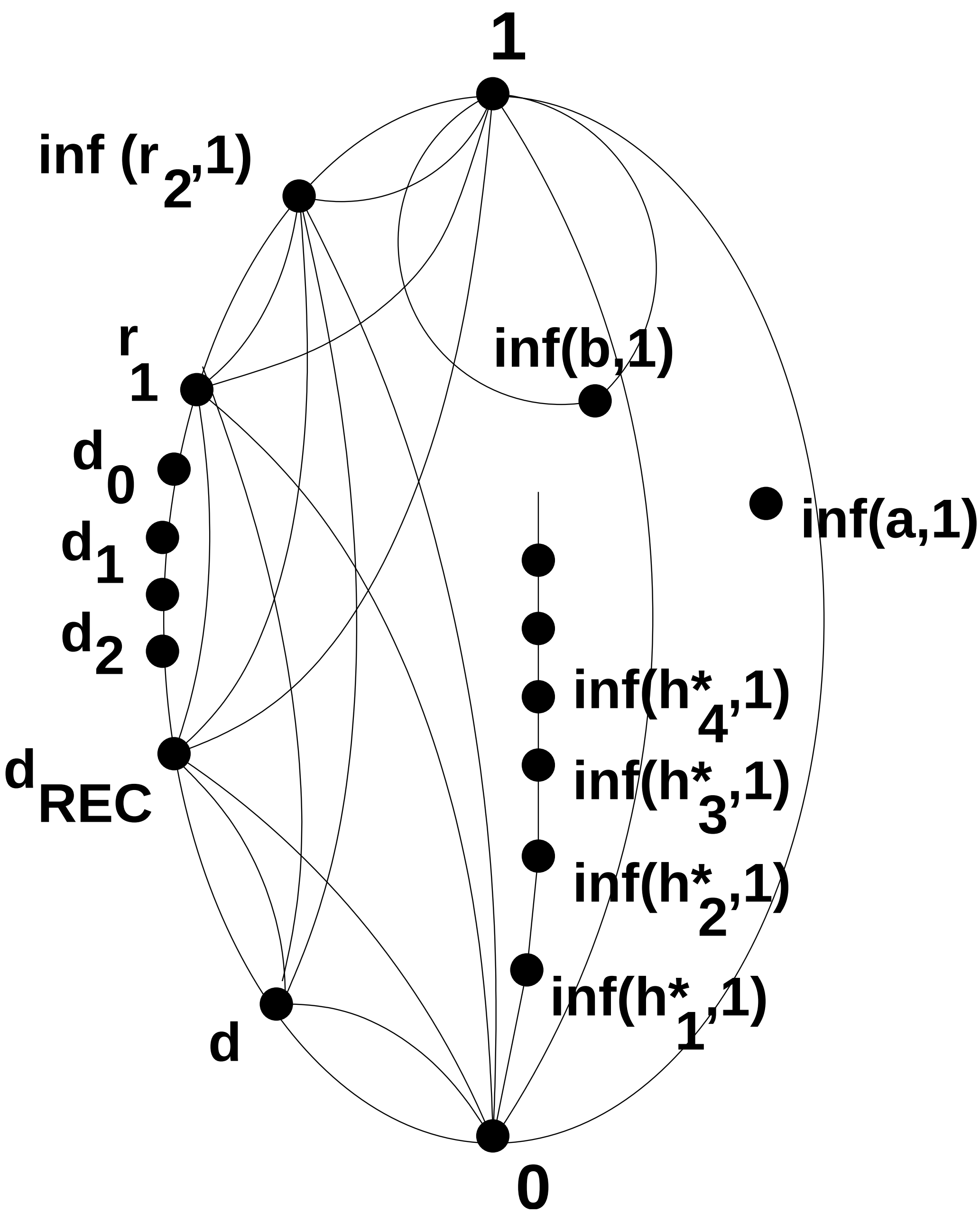
$\mathbf{d}_\alpha$ = the Muchnik degree of $\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is diagonally nonrecursive and α-*recursively bounded*$\}$ (bounded at level α of the Wainer hierarchy), $\alpha \leq \varepsilon_0$

$\mathbf{a}$ = the Muchnik degree of a recursively enumerable set

$\mathbf{h}_\alpha$ = the Muchnik degree of $0^{(\alpha)}$, $\alpha < \omega_1^{\mathsf{CK}}$

$\mathbf{h}_\alpha^*$ = the "blurred" version of $\mathbf{h}_\alpha$, $\alpha < \omega_1^{\mathsf{CK}}$

$\mathbf{b}$ = the Muchnik degree of $\{x \in \{0,1\}^{\mathbb{N}} \mid x$ is almost everywhere dominating$\}$



A picture of $\mathcal{P}_w$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{h} =$ hyperarithmeticity, $\mathbf{r} =$ randomness, $\mathbf{b} =$ almost everywhere domination, $\mathbf{d} =$ diagonal nonrecursiveness.

By Theorem 1, each of the black dots in the above picture is the Muchnik degree of a 2-dimensional subshift of finite type.

Thus we have apparently uncovered some interesting classes of 2-dimensional subshifts of finite type.

A basic result concerning $\mathcal{P}_w$ is as follows:

Embedding Lemma (Simpson 2004)**.** Let $\mathbf{s}$ be the Muchnik degree of a Σ^0_3 set. Then $\inf(\mathbf{s}, \mathbf{1})$ belongs to $\mathcal{P}_w$.

Combining this with Theorem 1, we obtain:

Theorem 4 (Simpson 2007)**.** Let $\mathbf{s}$ be the Muchnik degree of a Σ^0_3 set. Then there exists a 2-dimensional subshift of finite type whose Muchnik degree is $\inf(\mathbf{s}, \mathbf{1})$.

References:

Stephen Binns. A splitting theorem for the Medvedev and Muchnik lattices. *Mathematical Logic Quarterly*, 49:327–335, 2003.

Stephen Binns and Stephen G. Simpson. Embeddings into the Medvedev and Muchnik lattices of Π^0_1 classes. *Archive for Mathematical Logic*, 43:399–414, 2004.

Joshua A. Cole and Stephen G. Simpson. Mass problems and hyperarithmeticity. Preprint, 20 pages, 28 November 2006, submitted for publication.

Albert A. Muchnik. On strong and weak reducibilities of algorithmic problems. *Sibirskii Matematicheskii Zhurnal*, 4:1328–1341, 1963. In Russian.

Stephen G. Simpson. Medvedev degrees of 2-dimensional subshifts of finite type. *Ergodic Theory and Dynamical Systems*, to appear. Preprint, 8 pages, 1 May 2007.

Stephen G. Simpson. Mass problems and randomness. *Bulletin of Symbolic Logic*, 11:1–27, 2005.

Stephen G. Simpson. An extension of the recursively enumerable Turing degrees. *Journal of the London Mathematical Society*, 75:287–297, 2007.

Stephen G. Simpson. Mass problems and almost everywhere domination. *Mathematical Logic Quarterly*, 53:483–492, 2007.

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