

# Reverse mathematics, ordinal numbers, and the ACC

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## Basis theorems in algebra.

**Definition.** A ring satisfies the ACC (ascending chain condition) if every ascending sequence of ideals is finite.

Equivalently, every ideal is finitely generated.

**Theorem** (Hilbert 1890). Let  $K$  be a field. For all  $d$  the polynomial ring  $K[x_1, \dots, x_d]$  satisfies the ACC.

This is the Hilbert Basis Theorem.

The Hilbert Basis Theorem is very important in invariant theory and in algebraic geometry.

It is not to be confused with  
“basis theorems” in recursion theory!

## Basis theorems in algebra (continued).

There are also the following theorems.

**Theorem** (Hilbert ??). Let  $K$  be a field. For all  $d$  the formal power series ring  $K[[x_1, \dots, x_d]]$  satisfies the ACC.

**Theorem** (Robson 1978). Let  $K$  be a field. For all  $d$  the polynomial ring  $K\langle x_1, \dots, x_d \rangle$  in  $d$  *noncommuting* indeterminates satisfies the ACC for *insertive* ideals.

**Theorem** (Formanek/Lawrence 1976). Let  $K$  be a field of characteristic 0. Let  $S$  be the group of finitely supported permutations of  $\mathbb{N}$ . Then, the group ring  $K[S]$  satisfies the ACC.

## Some reverse mathematics.

Working in  $\text{RCA}_0$ , we restrict ourselves to countable fields.

**Theorem** (Simpson 1988). Over  $\text{RCA}_0$ ,

1. Hilbert's Theorem  $\iff \text{WO}(\omega^\omega)$ .
2. Robson's Theorem  $\iff \text{WO}(\omega^{\omega^\omega})$ .

**Theorem** (Hatzikiriakou 1994). Over  $\text{RCA}_0$ , Hilbert's Theorem for power series rings is equivalent to  $\text{WO}(\omega^\omega)$ .

**Theorem** (Hatzikiriakou/Simpson 2014). Over  $\text{RCA}_0$  the Formanek/Lawrence Theorem is equivalent to  $\text{WO}(\omega^\omega)$ .

Note: Hilbert's Theorem refers to an infinite sequence of rings, while Formanek/Lawrence refers to only one ring,  $K[S]$ .

We also show that, in all of these reverse mathematics results, the base theory  $\text{RCA}_0$  can be weakened to  $\text{RCA}_0^*$ .

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I will now give some details about the Formanek/Lawrence Theorem and its reversal.

## Partition theory.

As noted by Formanek and Lawrence, ideals in  $K[S]$  are in 1-to-1 correspondence with certain sets of partitions.

A partition of  $n$  is a finite sequence of integers  $n_1 \geq \cdots \geq n_k > 0$  such that  $n = n_1 + \cdots + n_k$ .

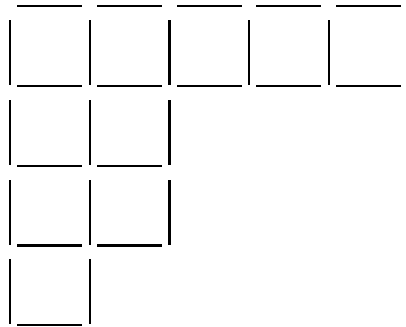
Example:  $(5, 2, 2, 1)$  is a partition of 10, because  $10 = 5 + 2 + 2 + 1$  and  $5 \geq 2 \geq 2 \geq 1 > 0$ .

Partitions of  $n$  are in 1-to-1 correspondence with conjugacy classes of  $S_n$ . Here  $S_n$  is the group of permutations of the set  $\{1, \dots, n\}$ .

Partition theory is a large branch of mathematics, closely connected to the representation theory of  $S_n$ .

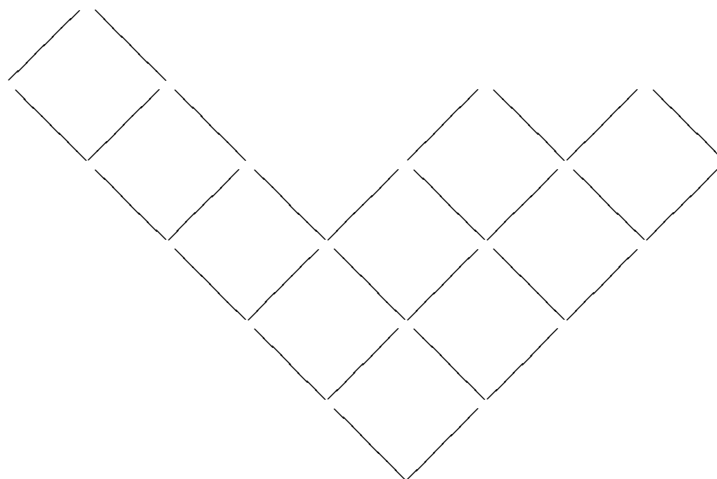
Note:  $S = \bigcup_{n=1}^{\infty} S_n$  and  $K[S] = \bigcup_{n=1}^{\infty} K[S_n]$ .

Partitions are often visualized as Young diagrams. For example, the partition  $10 = 5 + 2 + 2 + 1$  corresponds to the diagram



consisting of 10 boxes.

Rotated counterclockwise 135 degrees, it becomes a downwardly closed set in  $(\mathbb{N}^2, \leq)$  where  $(m, n) \leq (p, q) \iff (m \leq p \text{ and } n \leq q)$ .



A diagram is a finite downwardly closed set in  $\mathbb{N}^2$ . Let  $\mathcal{D}_2$  be the set of diagrams, partially ordered by inclusion.

A poset  $P$  is said to be WPO (well partially ordered) if  $(\forall f : \mathbb{N} \rightarrow P) \exists i \exists j (i < j \text{ and } f(i) \leq f(j))$ .

In my 1988 paper I show that, over  $\text{RCA}_0$ ,

1.  $K[x_1, \dots, x_d]$  has ACC  $\iff \mathbb{N}^d$  is WPO.
2.  $K\langle x_1, \dots, x_d \rangle$  ACC  $\iff \{x_1, \dots, x_d\}^*$  WPO.

Since  $\mathbb{N}^2$  is WPO, it follows by Higman's Lemma that  $\mathcal{D}_2$  is WPO.

A set  $\mathcal{U} \subseteq \mathcal{D}_2$  is said to be closed if  $\forall D (D \in \mathcal{U} \iff \forall E (D \subset E \Rightarrow E \in \mathcal{U}))$ .

This implies that  $\mathcal{U}$  is upwardly closed, but not conversely!

Formanek and Lawrence exhibit a 1-to-1 correspondence between ideals in  $K[S]$  and closed sets in  $\mathcal{D}_2$ . Since  $\mathcal{D}_2$  is WPO, it follows that  $\mathcal{D}_2$  has the ACC on closed sets, hence  $K[S]$  has the ACC on 2-sided ideals.



## Reversing Formanek/Lawrence.

Working in  $\text{RCA}_0$ , we formalize the work of Formanek/Lawrence to prove that  $K[S]$  has ACC if and only if  $\mathcal{D}_2$  has the ACC on closed sets. Also working in  $\text{RCA}_0$ , we use methods of Simpson 1988 to prove that  $\text{WO}(\omega^\omega) \iff \mathcal{D}_2$  is WPO.

Still working in  $\text{RCA}_0$ , it remains to prove:  $\mathcal{D}_2$  is WPO  $\iff \mathcal{D}_2$  has ACC on closed sets. To prove this, we use a new combinatorial lemma.

**Lemma.** Let  $\mathcal{S}$  be a finite set of diagrams. Then, the closure of  $\mathcal{S}$  is equal to the upward closure of  $\{D_0 \cup E_1 \mid D, E \in \mathcal{S}\}$ . Moreover, there are only finitely many diagrams in the closure of  $\mathcal{S}$  which are not in the upward closure of  $\mathcal{S}$ .

$D_0$  and  $D_1$  are the results of truncating the first row and first column of  $D$ , respectively.

For example, if  $D = (5, 2, 2, 1)$  then  $D_0 = (2, 2, 2, 1)$  and  $D_1 = (5, 2, 2)$ .

## Weakening the base theory.

Recall that  $\text{RCA}_0^*$  is  $\text{RCA}_0$  minus  $\Sigma_1^0$  induction.  $\text{RCA}_0^*$  is proof-theoretically weaker than  $\text{RCA}_0$ .

In our ACC reversals, we wish to replace  $\text{RCA}_0$  by  $\text{RCA}_0^*$ . For this, it suffices to prove in  $\text{RCA}_0^*$  that if  $K[x]$  has ACC then  $\Sigma_1^0$  induction holds.

**Lemma.** Over  $\text{RCA}_0^*$ , if  $\Sigma_1^0$  induction fails, then there exists  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  
(1)  $f(i) \geq f(i + 1)$  for all  $i \in \mathbb{N}$ , and  
(2)  $f(i) > f(i + 1)$  for infinitely many  $i \in \mathbb{N}$ .

In this situation, letting  $n_i = 2^{f(i)}$ , the ideals in  $K[x]$  generated by  $x^{n_i}$  for each  $i \in \mathbb{N}$  are a counterexample to the ACC.

## Philosophical aspect.

Recently I suggested that, in contrast to the concept of potential infinity, the concept of actual infinity appears to lack objective justification. Therefore, in order to promote objectivity in mathematics, it seems desirable to limit the use of actual infinity.

I see a close connection to Hilbert's program of finitistic reductionism. Let us say that a system  $T \subseteq Z_2$  is finitistically reducible if all  $\Pi_1^0$  (or possibly even  $\Pi_2^0$ ) sentences provable in  $T$  are provable in PRA, i.e., Primitive Recursive Arithmetic.

Some important systems *are* finitistically reducible, namely  $WKL_0$ , and  $WKL_0 + \Sigma_2^0$  bounding, and some stronger systems.

On the other hand,  $RCA_0 + WO(\omega^\omega)$  and  $RCA_0 + \Sigma_2^0$  induction *are not* finitistically reducible, because they prove  $\text{Con}(\text{PRA})$  and totality of the Ackermann function.

## **Philosophical aspect (continued).**

In particular, the Hilbert Basis Theorem and the Formanek/Lawrence Theorem *are not* finitistically reducible.

(However, for each specific positive integer  $d$ , the Hilbert Basis Theorem for  $K[x_1, \dots, x_d]$  *is* finitistically reducible, since provable in  $\text{RCA}_0$ .)

Recently Chong, Slaman, Yang, and Yokoyama have done some important work on the reverse mathematics of  $\text{RT}(2, 2)$ , i.e., Ramsey's Theorem for exponent 2.

**An important open question remains:**

Is  $\text{RCA}_0 + \text{RT}(2, 2)$  finitistically reducible?

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## **Thank you for your attention!**