Reverse mathematics, ordinal numbers, and the ACC

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In abstract algebra, a ring is said to satisfy the ACC (ascending chain condition) if it has no infinite ascending sequence of ideals. According to a famous and controversial theorem of Hilbert, 1890, polynomial rings with finitely many indeterminates satisfy the ACC. There is a similar theorem for noncommuting indeterminates, due to J. C. Robson, 1978. In 1988 I performed a reverse-mathematical analysis of the theorems of Hilbert and Robson, proving that they are equivalent over RCA_0 to the well-orderedness of (the standard notation systems for) the ordinal numbers ω^ω and ω^{ω^ω} respectively. Now I perform a similar analysis of a theorem of Formanek and Lawrence, 1976. Let S be the group of finitely supported permutations of the natural numbers. Let $K[S]$ be the group ring of S over a countable field K of characteristic 0. Formanek and Lawrence proved that $K[S]$ satisfies the ACC. All of these results concerning the ACC involve well partial ordering theory. I now prove that the Formanek/Lawrence theorem is equivalent over RCA_0 to the well-orderedness of ω^ω. The proof involves an apparently new, combinatorial lemma concerning Young diagrams. I also show that, in all of these reverse-mathematical results, RCA_0 can be weakened to RCA_0^*. This recent work was done jointly with Kostas Hatzikiriakou.

In addition, I make some remarks concerning reverse mathematics as it applies to Hilbert's foundational program of finitistic reductionism. It is significant that RCA_0 and WKL_0 and even $\mathrm{WKL}_0 + \Sigma^0_2$-bounding are Π^0_2-conservative over PRA, while Σ_2-induction and the well-orderedness of ω^ω are not even Π^0_1-conservative over PRA. The proof-theoretic strength of RCA_0 + Ramsey's Theorem for exponent 2 is an open question which has attracted much recent attention.