The Gödel Hierarchy and Reverse Mathematics

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Of the 23 Hilbert problems, 1 and 2 belong to foundations of mathematics, while 10 and 17 are closely related to mathematical logic. In addition, from Professor Bottazzini's talk at this conference, I learned that problems 3, 4, 5 were an outgrowth of Hilbert's interest in foundations of geometry.

Hilbert's 1900 Problem List

1. Cantor's Problem of the Cardinal Number of the Continuum.

2. Compatibility of the Arithmetical Axioms.

. . .

10. Determination of the Solvability of a Diophantine Equation.

. . .

17. Expression of Definite Forms by Squares.

. . .

Gödel's Incompleteness Theorem:

Hilbert's concern for consistency proofs led to Gödel's Second Incompleteness Theorem.

Let T be a theory in the predicate calculus, satisfying certain mild conditions. Then:

1. T is incomplete.

2. The statement "T is consistent" is not a theorem of T.

(Gödel 1931)

3. The problem of deciding whether a given formula is a theorem of T is algorithmically unsolvable.

(Gödel, Turing, Tarski, . . .)

Some people believe that Gödel's Incompleteness Theorem means the end of the axiomatic method generally, and of Hilbert's Program specifically.

In my opinion, this view fails to take account of f.o.m. developments subsequent to 1931. The purpose of this talk is to outline some relatively recent research which reveals logical regularity and structure arising from the axiomatic approach to f.o.m.

1. The Gödel Hierarchy

2. Reverse Mathematics

3. Foundational consequences of R. M.

4. A partial realization of Hilbert's Program

The Gödel Hierarchy:

Let T_1, T_2 be two theories as above. Define

$$T_1 \ < \ T_2$$

if "T_1 is consistent" is a theorem of T_2.

Usually this is equivalent to saying that T_1 is interpretable in T_2 and not vice versa.

This ordering gives a hierarchy of foundational theories, the Gödel Hierarchy.

The Gödel Hierarchy is linear and exhibits other remarkable regularities.

The Gödel Hierarchy is a central object of study in foundations of mathematics.

Stopping Points in the Gödel Hierarchy:

strong:
- ⋮
- supercompact cardinal
- ⋮
- measurable cardinal
- ⋮
- ZFC (ZF set theory with choice)
- Zermelo set theory
- simple type theory

medium:
- Z_2 (2nd order arithmetic)
- ⋮
- Π^1_2 comprehension
- Π^1_1 comprehension
- ATR_0 (arith. transfinite recursion)
- ACA_0 (arithmetical comprehension)

weak:
- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- EFA (elementary arithmetic)
- bounded arithmetic
- ⋮

Foundations of mathematics (f.o.m.):

Foundations of mathematics is the study of the most basic concepts and logical structure of mathematics as a whole.

Among the most basic mathematical concepts are: number, shape, set, function, algorithm, mathematical proof, mathematical definition, mathematical axiom, mathematical theorem, mathematical statement.

A key f.o.m. question:

What are the appropriate axioms for mathematics?

The FOM mailing list:

FOM is an automated e-mail list for discussing foundations of mathematics. There are currently more than 500 subscribers. There have been more than 5000 postings.

FOM is maintained and moderated by S. Simpson. The FOM Editorial Board consists of M. Davis, H. Friedman, C. Jockusch, D. Marker, S. Simpson, A. Urquhart.

FOM postings and information are available on the web at

`http://www.math.psu.edu/simpson/fom/`

Friedman and Simpson founded FOM in order to promote a controversial idea: **mathematical logic is or ought to be driven by f.o.m. considerations.**

f.o.m. = foundations of mathematics.

Background of Reverse Mathematics:

Second order arithmetic (Z_2) is a two-sorted system.

Number variables $m, n, \ldots$ range over

$$\omega = \{0, 1, 2, \ldots\} .$$

Set variables $X, Y, \ldots$ range over subsets of ω.

We have $+$, $\times$, $=$ on ω, plus the membership relation

$$\in \quad = \quad \{(n, X) : n \in X\} \quad \subseteq \quad \omega \times P(\omega) .$$

Within subsystems of second order arithmetic, we can formalize rigorous mathematics (analysis, algebra, geometry, . . .).

Subsystems of second order arithmetic are basic to our current understanding of the logical structure of contemporary mathematics.

An essential reference:

David Hilbert and Paul Bernays
Grundlagen der Mathematik
("Foundations of Mathematics")
Second Edition
Volume I, XV + 475 pages
Volume II, XIV + 561 pages
Grundlehren der Mathematischen Wißenschaften
Springer-Verlag, 1968–1970

In Supplement IV ("Appendix IV") of *Grundlagen der Mathematik*, Hilbert and Bernays present the formalization of rigorous mathematics within second order arithmetic ($= Z_2$).

Themes of Reverse Mathematics:

Reverse Mathematics is a research program developed by H. Friedman, S. Simpson, Simpson's students, and other researchers.

Let τ be a mathematical theorem. Let S_τ be the weakest natural subsystem of second order arithmetic in which τ is provable.

1. Very often, the principal axiom of S_τ is logically equivalent to τ.

2. Furthermore, only a few subsystems of second order arithmetic arise in this way.

For a full exposition, see my book.

Book on Reverse Mathematics:

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Perspectives in Mathematical Logic
Springer-Verlag, 1998
XIV + 445 pages

`http://www.math.psu.edu/simpson/sosoa/`

Order: 1-800-SPRINGER

List price: $60

Discount: 30 percent for ASL members, mention promotion code S206

Unfortunately the book is no longer available from Springer, but there is hope that the ASL will reprint it soon.

Another book on Reverse Mathematics:

S. G. Simpson (editor)
Reverse Mathematics 2001

A volume of papers by various authors,
to appear in 2001,
approximately 400 pages.

http://www.math.psu.edu/simpson/revmath/

The Big Five:

For Reverse Mathematics, the five most important subsystems of Z_2 are:

$\mathsf{RCA}_0 =$ formalized computable mathematics (Recursive Comprehension Axiom)

$\mathsf{WKL}_0 = \mathsf{RCA}_0 +$ a compactness principle (Weak König's Lemma)

$\mathsf{ACA}_0 = \mathsf{RCA}_0 +$ the Turing jump operator (Arithmetical Comprehension Axiom)

$\mathsf{ATR}_0 = \mathsf{ACA}_0 +$ transfinite recursion (Arithmetical Transfinite Recursion)

$\Pi^1_1\text{-}\mathsf{CA}_0 = \mathsf{ACA}_0 + \Pi^1_1$ comprehension (Π^1_1 Comprehension Axiom)

Themes of R. M. (continued):

We develop a table indicating which mathematical theorems can be proved in which subsystems of Z_2.

	RCA_0	WKL_0	ACA_0	ATR_0	Π^1_1-CA_0
analysis (separable):					
differential equations	X	X			
continuous functions	X, X	X, X	X		
completeness, *etc.*	X	X	X		
Banach spaces	X	X, X			X
open and closed sets	X	X		X, X	X
Borel and analytic sets	X			X, X	X, X
algebra (countable):					
countable fields	X	X, X	X		
commutative rings	X	X	X		
vector spaces	X		X		
Abelian groups	X		X	X	X
miscellaneous:					
mathematical logic	X	X			
countable ordinals	X		X	X, X	
infinite matchings		X	X	X	
the Ramsey property			X	X	X
infinite games			X	X	X

Reverse Mathematics for WKL_0:

WKL_0 is equivalent over RCA_0 to each of the following mathematical statements:

1.The Heine/Borel Covering Lemma: Every covering of $[0, 1]$ by a sequence of open intervals has a finite subcovering.

2. Every covering of a compact metric space by a sequence of open sets has a finite subcovering.

3. Every continuous real-valued function on $[0, 1]$ (or on any compact metric space) is bounded (uniformly continuous, Riemann integrable).

6. The Maximum Principle: Every continuous real-valued function on $[0, 1]$ (or on any compact metric space) has (or attains) a supremum.

R. M. for WKL_0 (continued):

7. The local existence theorem for solutions of (finite systems of) ordinary differential equations.

8. Gödel's Completeness Theorem: every finite (or countable) set of sentences in the predicate calculus has a countable model.

9. Every countable commutative ring has a prime ideal.

10. Every countable field (of characteristic 0) has a unique algebraic closure.

11. Every countable formally real field is orderable.

12. Every countable formally real field has a (unique) real closure.

R. M. for WKL_0 (continued):

13. Brouwer's Fixed Point Theorem: Every (uniformly) continuous function $\phi : [0,1]^n \to [0,1]^n$ has a fixed point.

14. The Separable Hahn/Banach Theorem: If f is a bounded linear functional on a subspace of a separable Banach space, and if $\|f\| \leq 1$, then f has an extension $\widetilde{f}$ to the whole space such that $\|\widetilde{f}\| \leq 1$.

15. Banach's Theorem: In a separable Banach space, given two disjoint convex open sets A and B, there exists a closed hyperplane H such that A is on one side of H and B is on the other.

16. Every countable k-regular bipartite graph has a perfect matching.

Reverse Mathematics for ACA_0

ACA_0 is equivalent over RCA_0 to each of the following mathematical statements:

1. Every bounded, or bounded increasing, sequence of real numbers has a least upper bound.

2. The Bolzano/Weierstraß Theorem: Every bounded sequence of real numbers, or of points in $\mathbb{R}^n$, has a convergent subsequence.

3. Every sequence of points in a compact metric space has a convergent subsequence.

4. The Ascoli Lemma: Every bounded equicontinuous sequence of real-valued continuous functions on a bounded interval has a uniformly convergent subsequence.

5. Every countable commutative ring has a maximal ideal.

R. M. for ACA_0 (continued):

6. Every countable vector space over $\mathbb{Q}$, or over any countable field, has a basis.

7. Every countable field (of characteristic 0) has a transcendence basis.

8. Every countable Abelian group has a unique divisible closure.

9. König's Lemma: Every infinite, finitely branching tree has an infinite path.

10. Ramsey's Theorem for colorings of $[\mathbb{N}]^3$, or of $[\mathbb{N}]^4$, $[\mathbb{N}]^5$, ...

Reverse Mathematics for ATR_0:

ATR_0 is equivalent over RCA_0 to each of the following mathematical statements:

1. Any two countable well orderings are comparable.

2. Ulm's Theorem: Any two countable reduced Abelian p-groups which have the same Ulm invariants are isomorphic.

3. The Perfect Set Theorem: Every uncountable closed, or analytic, set has a perfect subset.

4. Lusin's Separation Theorem: Any two disjoint analytic sets can be separated by a Borel set.

R. M. for ATR_0 (continued):

5. The domain of any single-valued Borel set in the plane is a Borel set.

6. Every clopen (or open) game in $\mathbb{N}^{\mathbb{N}}$ is determined.

7. Every clopen (or open) subset of $[\mathbb{N}]^{\mathbb{N}}$ has the Ramsey property.

8. Every countable bipartite graph admits a König covering.

Reverse Mathematics for Π^1_1-CA_0

Π^1_1-CA_0 is equivalent over RCA_0 to each of the following mathematical statements:

1. Every tree has a largest perfect subtree.

2. The Cantor/Bendixson Theorem: Every closed subset of $\mathbb{R}$, or of any complete separable metric space, is the union of a countable set and a perfect set.

3. Every countable Abelian group is the direct sum of a divisible group and a reduced group.

4. Every difference of two open sets in the Baire space $\mathbb{N}^{\mathbb{N}}$ is determined.

5. Every G_δ set in $[\mathbb{N}]^{\mathbb{N}}$ has the Ramsey property.

R. M. for Π^1_1-CA_0 (continued):

6. Silver's Theorem: For every Borel (or co-analytic, or F_σ) equivalence relation with uncountably many equivalence classes, there exists a perfect set of inequivalent elements.

7. For every countable set S in the dual X^* of a separable Banach space X (or in $\ell_1 = c_0^*$), there exists a smallest weak-$*$-closed subspace of X^* (or of ℓ_1) containing S.

8. For every norm-closed subspace Y of $\ell_1 = c_0^*$, the weak-$*$-closure of Y exists.

My Reverse Mathematics Ph.D. students:

1. John Steel, *Determinateness and Subsystems of Analysis*, University of California at Berkeley, 1977.

2. Rick L. Smith, *Theory of Profinite Groups with Effective Presentations*, Pennsylvania State University, 1979.

5. Stephen H. Brackin, *On Ramsey-type Theorems and their Provability in Weak Formal Systems*, Pennsylvania State University, 1984.

7. Douglas K. Brown, *Functional Analysis in Weak Subsystems of Second Order Arithmetic*, Pennsylvania State University, 1987.

8. Jeffry L. Hirst, *Combinatorics in Subsystems of Second Order Arithmetic*, Pennsylvania State University, 1987.

9. Xiaokang Yu (Connie Yu), *Measure Theory in Weak Subsystems of Second Order Arithmetic*, Pennsylvania State University, 1987.

10. Fernando Ferreira, *Polynomial Time Computable Arithmetic and Conservative Extensions*, Pennsylvania State University, 1988.

11. Kostas Hatzikiriakou, *Commutative Algebra in Subsystems of Second Order Arithmetic*, Pennsylvania State University, 1989.

12. Alberto Marcone, *Foundations of BQO Theory and Subsystems of Second Order Arithmetic*, Pennsylvania State University, 1993.

13. A. James Humphreys, *On the Necessary Use of Strong Set Existence Axioms in Analysis and Functional Analysis*, Pennsylvania State University, 1996.

14. Mariagnese Giusto, *Topology, Analysis and Reverse Mathematics*, University of Torino, 1998.

Foundational consequences of R. M.:

1. We precisely classify mathematical theorems, according to which subsystems of Z_2 they are provable in.

2. We identify certain subsystems of Z_2 as being mathematically natural.
   The naturalness is rigorously demonstrated.

3. We work out the consequences of particular foundational doctrines:

   - recursive analysis (Pour-El/Richards)
   - constructivism (Bishop)
   - finitistic reductionism (Hilbert)
   - predicativity (Weyl)
   - predicative reductionism (Feferman/Friedman/Simpson)
   - impredicative analysis (Takeuti/Schütte/Pohlers)

Foundational consequences (continued):

By means of Reverse Mathematics, we identify five particular subsystems of Z_2 as being mathematically natural. We correlate these systems to traditional f.o.m. programs.

RCA_0	constructivism	Bishop
WKL_0	finitistic reductionism	Hilbert
ACA_0	predicativity	Weyl, Feferman
ATR_0	predicative reductionism	Friedman, Simpson
Π^1_1-CA_0	impredicativity	Feferman *et al.*

We analyze these f.o.m. programs in terms of their consequences for mathematical practice. Specifically, under the various proposals, which mathematical theorems are "lost"? Reverse Mathematics provides precise answers to such questions.

Hilbert's Program:

Hilbert 1925 proposed to reduce all of mathematics to finitistic mathematics.

Gödel's Theorem implies that Hilbert's Program cannot be completely realized. For instance, "finitism is consistent" is finitistically meaningful yet not finitistically provable.

Nevertheless, a significant partial realization of Hilbert's Program has been obtained.

1. W. W. Tait has argued that PRA embodies finitism.

2. H. Friedman has shown that WKL_0 is conservative over PRA for Π^0_2 sentences. This class includes all finitistically meaningful sentences.

3. A large portion of core mathematics can be carried out in WKL_0, including many of the best known nonconstructive theorems.

This is a byproduct of Reverse Mathematics.

Additional references:

Stephen G. Simpson
Partial realizations of Hilbert's Program
Journal of Symbolic Logic, volume 53, 1988
pages 349-363

This paper and others are available at

`http://www.math.psu.edu/simpson/papers/.`

Also available there is a detailed summary of my book:

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Springer-Verlag, 1998, XIV + 445 pages

THE END