Recent Aspects of Mass Problems: Symbolic Dynamics and Intuitionism

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Mathematical Research Institute
Oberwolfach, Germany
April 6–12, 2008

Abstract.

Mass Problems:

A set $P \subseteq \{0,1\}^{\mathbb{N}}$ may be viewed as a *mass problem*, i.e., a decision problem with more than one solution. By definition, the *solutions* of P are the elements of P. A mass problem is said to be *solvable* if at least one of its solutions is recursive. A mass problem P is said to be *weakly reducible* to a mass problem Q if for each solution of Q there exists a solution of P which is Turing reducible to the given solution of Q. A *weak degree* is an equivalence class of mass problems under mutual weak reducibility. The lattice $\mathcal{D}_w$ of all weak degrees is due to Muchnik 1963. There is an obvious embedding of the Turing degrees into $\mathcal{D}_w$.

A set $P \subseteq \{0,1\}^{\mathbb{N}}$ is said to be Π^0_1 if it is *effectively closed*, i.e., it is the complement of the union of a recursive sequence of basic open sets. Let $\mathcal{P}_w$ denote the sublattice of $\mathcal{D}_w$ consisting of the mass problems associated with nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$. The lattice $\mathcal{P}_w$ has been investigated by Simpson and his collaborators. There is a non-obvious but natural embedding of the recursively enumerable Turing degrees into $\mathcal{P}_w$. It is known that $\mathcal{P}_w$ contains many specific, natural weak degrees which are related to various topics in the foundations of mathematics. Among these topics are reverse mathematics, algorithmic randomness, Kolmogorov complexity, almost everywhere domination, hyperarithmeticity, effective Hausdorff dimension, resource-bounded computational complexity, and subrecursive hierarchies.

Symbolic Dynamics:

Let A be a finite set of symbols. The *full two-dimensional shift* on A is the dynamical system consisting of the natural action of the group $\mathbb{Z} \times \mathbb{Z}$ on the compact space $A^{\mathbb{Z}\times\mathbb{Z}}$. A *two-dimensional subshift* is a nonempty closed subset of $A^{\mathbb{Z}\times\mathbb{Z}}$ which is invariant under the action of $\mathbb{Z} \times \mathbb{Z}$. A two-dimensional subshift is said to

Abstract (continued):

be *of finite type* if it is defined by a finite set of excluded configurations. The two-dimensional subshifts of finite type are known to form an important class of dynamical systems, with connections to mathematical physics, etc.

Clearly every two-dimensional subshift of finite type is a nonempty Π^0_1 subset of $A^{\mathbb{Z}\times\mathbb{Z}}$, hence its weak degree belongs to $\mathcal{P}_w$. Conversely, we prove that every weak degree in $\mathcal{P}_w$ is the weak degree of a two-dimensional subshift of finite type. The proof of this result uses tilings of the plane. We present an application of this result to symbolic dynamics. Namely, we obtain an infinite family of two-dimensional subshifts of finite type which are, in a certain sense, mutually incompatible. Our application is stated purely in terms of two-dimensional subshifts of finite type, with no mention of weak degrees.

Intuitionism:

Historically, the study of mass problems originated from intuitionistic considerations. Kolmogorov 1932 proposed to view intuitionism as a "calculus of problems." Muchnik 1963 introduced weak degrees as a rigorous elaboration of Kolmogorov's proposal. As noted by Muchnik, the lattice $\mathcal{D}_w$ of all weak degrees is Brouwerian.

The question arises, is the sublattice $\mathcal{P}_w$ Brouwerian? We prove that it is not. The proof uses our adaptation of a technique of Posner and Robinson.

Part 1: SYMBOLIC DYNAMICS

We begin with the 1-dimensional case.

A *dynamical system* consists of
a nonempty set X (the set of *states*)
plus a mapping $T : X \to X$
(the *state transition operator*).

Throughout this talk we assume that X is compact and metrizable. We also assume that T is continuous, one-to-one, and onto.

Example. Let A be a finite set of symbols. $A^{\mathbb{Z}}$ is the set of bi-infinite sequences of symbols from A. The *shift operator* $S : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is given by $S(x)(n) = x(n+1)$ for all $x \in A^{\mathbb{Z}}$.

The dynamical system consisting of the compact metrizable space $A^{\mathbb{Z}}$ and the shift operator S is known as the *full shift* on A.

Let X be a nonempty closed subset of $A^{\mathbb{Z}}$ which is *invariant under the shift operator*, i.e., $x \in X \iff S(x) \in X$ for all x.

The dynamical system consisting of the compact metrizable space X together with the shift operator S (actually $S \restriction X$)
is known as a *subshift* on A.
It is a subsystem of the full shift on A.

There are many different kinds of subshifts. Subshifts are very useful for describing the behavior of dynamical systems in general.

The study of subshifts for their own sake is called *symbolic dynamics*.

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Every subshift $X \subseteq A^{\mathbb{Z}}$ is defined by a set of *excluded words*. Namely, for an appropriate set E of finite sequences of symbols from A,

$$X = \{x \in A^{\mathbb{Z}} \mid x \text{ contains no consecutive subsequence belonging to } E\}.$$

If E is finite, we say that X is *of finite type*.

Subshifts of finite type have been studied extensively. It is easy to see that every 1-dimensional subshift of finite type contains periodic points.

Note: E is recursive if and only if X is Π^0_1.

Cenzer/Dashti/King 2007 have constructed a 1-dimensional Π^0_1 subshift which contains no recursive points, hence no periodic points.

Many questions regarding 1-dimensional Π^0_1 subshifts remain open.

We now turn to the 2-dimensional case.

As before, let A be a finite set of symbols.

Let $A^{\mathbb{Z}\times\mathbb{Z}}$ be the set of doubly bi-infinite double sequences of symbols from A.

This is again a compact metrizable space.

Points of $A^{\mathbb{Z}\times\mathbb{Z}}$ may be viewed as *tilings of the plane*, in the sense of Wang 1961.

Tiling problems were studied by logicians during the years 1960–1980.

The connection with dynamical systems was noticed only relatively recently.

A *2-dimensional dynamical system* consists of a nonempty set X and a commuting pair of maps $T_1, T_2 : X \to X$. As before we assume X compact metrizable, T_1, T_2 continuous one-to-one onto.

The *full 2-dimensional shift* on A is the dynamical system consisting of $A^{\mathbb{Z}\times\mathbb{Z}}$ with shift operators $S_1, S_2 : A^{\mathbb{Z}\times\mathbb{Z}} \to A^{\mathbb{Z}\times\mathbb{Z}}$ given by $S_1(x)(m,n) = x(m+1,n)$ and $S_2(x)(m,n) = x(m,n+1)$.

A *2-dimensional subshift* on A is
a nonempty closed set $X \subseteq A^{\mathbb{Z}\times\mathbb{Z}}$
which is invariant under S_1 and S_2.

Note that (X, S_1, S_2) is again a 2-dimensional dynamical system. It is a subsystem of the full 2-dimensional shift on A.

As in the 1-dimensional case, every 2-dimensional subshift X is defined by a set E of excluded configurations.

If E is finite, X is said to be *of finite type*.

Here, by a *configuration* we mean a "2-dimensional word," i.e., a member of $A^{\{1,\ldots,r\}\times\{1,\ldots,r\}}$ for some positive integer r.

2-dimensional subshifts of finite type are important in dynamical systems theory. An example is the Ising model in mathematical physics.

History:

Berger 1966 answered a question of Wang 1961 by constructing a 2-dimensional subshift of finite type with no periodic points.

Berger 1966 showed that it is undecidable whether a given finite set of excluded configurations defines a (nonempty!) 2-dimensional subshift.

Myers 1974 constructed a 2-dimensional subshift of finite type with no recursive points.

Hochman/Meyerovitch 2007 proved: a real number $h \geq 0$ is the entropy of a 2-dimensional subshift of finite type if and only if h is *right recursively enumerable*. This means that h is the limit of a recursive decreasing sequence of rational numbers.

Using the methods of Robinson 1971 and Myers 1974, I have proved:

Theorem 1 (Simpson 2007). The Muchnik degrees of 2-dimensional subshifts of finite type are the same as the Muchnik degrees of nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$.

This theorem is useful, because we can then apply known results from recursion theory to study 2-dimensional subshifts of finite type.

Below we shall present one such application.

Our application will be stated purely in terms of 2-dimensional subshifts of finite type, with no mention of Muchnik degrees and no mention of recursion theory.

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To state our application, we need some easy definitions which make perfect sense for all dynamical systems.

Definition. Let X and Y be 2-dimensional subshifts on k and l symbols respectively. The Cartesian product $X \times Y$ and the disjoint union $X + Y$ are 2-dimensional subshifts on kl and $k + l$ symbols respectively.

Definition. Let (X, S_1, S_2) be a 2-dimensional subshift on k symbols. Let a, b, c, d be integers with $ad - bc \neq 0$. Then, the system $(X, S_1^a S_2^b, S_1^c S_2^d)$ is canonically isomorphic to a 2-dimensional subshift on $k^{|ad-bc|}$ symbols.

Definition. If $\mathcal{U}$ is a set of 2-dimensional subshifts, let cl($\mathcal{U}$) be the closure of $\mathcal{U}$ under the above operations.

Definition. If X and Y are 2-dimensional subshifts, a *shift morphism* from X to Y is a continuous mapping $F : X \to Y$ which commutes with the shift operators.

In other words, $F(S_1(x)) = S_1(F(x))$ and $F(S_2(x)) = S_2(F(x))$ for all $x \in X$.

Now for the application.

Theorem 2 (Simpson 2007).
There is an infinite set $\mathcal{W}$ of 2-dimensional subshifts of finite type, such that for any partition $\mathcal{U}, \mathcal{V}$ of $\mathcal{W}$, and for any $X \in \mathrm{cl}(\mathcal{U})$ and $Y \in \mathrm{cl}(\mathcal{V})$, there is no shift morphism from X to Y or vice versa.

Theorem 2 follows from Theorem 1 plus a previously known recursion-theoretic result:

> There is an infinite set of nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ whose Muchnik degrees are independent.

This known recursion-theoretic result is proved by means of a priority argument.

We now discuss some ingredients of Theorems 1 and 2 and their proofs.

A subset of $A^{\mathbb{Z}\times\mathbb{Z}}$ or of $A^{\mathbb{Z}}$ or of $\{0,1\}^{\mathbb{N}}$ is *effectively closed* if it is the complement of the union of a recursive sequence of basic open sets. Here a *basic open set* is any set of the form $N_\sigma = \{x \mid x \restriction \mathrm{dom}(\sigma) = \sigma\}$ where σ is a finite partial function.

By definition, a set is Π^0_1 if and only if it is effectively closed.

The spaces $A^{\mathbb{Z}\times\mathbb{Z}}$ and $A^{\mathbb{Z}}$ and $\{0,1\}^{\mathbb{N}}$ are recursively homeomorphic to each other.

Hence, Π^0_1 sets in any of them are recursively homeomorphic to Π^0_1 sets in all of them.

Clearly subshifts of finite type are Π^0_1. More generally, any subshift defined by a recursive sequence of excluded configurations is Π^0_1.

Let X and Y be sets.

Y is *Muchnik reducible to* X if $(\forall x \in X)$ $(\exists$ partial recursive functional $F)\,(F(x) \in Y)$.

X and Y are *Muchnik equivalent* if each is Muchnik reducible to the other.

Recursively homeomorphic sets are Muchnik equivalent, but not conversely.

A *Muchnik degree* is an equivalence class under Muchnik equivalence.

The Muchnik degrees of all nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ under Muchnik reducibility form a distributive lattice, denoted $\mathcal{P}_w$.

It is known that $\mathcal{P}_w$ is structurally rich.

Theorem 1 says that the Muchnik degrees of 2-dimensional subshifts of finite type are precisely the Muchnik degrees in $\mathcal{P}_w$.

By contrast, all 1-dimensional subshifts of finite type are of Muchnik degree $\mathbf{0}$.

Thus, the 2-dimensional case is much more complicated than the 1-dimensional case.

Let X and Y be 2-dimensional subshifts.

A basic fact concerning shift morphisms:

Each shift morphism $F : X \to Y$ is describable in a very simple manner as a *block code*. This means that $F(x)(m,n)$ depends only on $x(m \pm i, n \pm j)$, $i,j \in \{0,\ldots,r\}$ for some fixed r.

In particular, each shift morphism is given by a recursive functional. Thus, the existence of a shift morphism from X to Y implies that Y is Muchnik reducible to X.

Define $X \geq Y$ if there exists a shift morphism from X to Y. Define $X \equiv Y$ if $X \geq Y$ and $Y \geq X$. The $\equiv$-equivalence classes form a distributive lattice. We have:

Theorem 3 (Simpson 2007). There is a canonical lattice homomorphism of the lattice of $\equiv$-equivalence classes of 2-dimensional subshifts of finite type, onto the lattice $\mathcal{P}_w$.

In all of these lattices, the supremum and infimum are given by $X \times Y$ and $X + Y$.

If X is a 2-dimensional subshift of finite type, there are surely some interesting relationships between the dynamical properties of X and the Muchnik degree of X.

These relationships remain to be explored.

Moreover $\mathcal{P}_w$ is structurally rich and contains many specific, natural degrees which are motivated by the idea of *mass problems* in recursion theory.

These specific, natural degrees in $\mathcal{P}_w$ are linked to foundational topics:

- algorithmic randomness
- reverse mathematics
- almost everywhere domination
- diagonal nonrecursiveness
- hyperarithmeticity
- resource-bounded computational complexity
- Kolmogorov complexity
- subrecursive hierarchies

Some examples of Muchnik degrees in $\mathcal{P}_w$:

$\mathbf{0}$ = the bottom degree in $\mathcal{P}_w$
= the Muchnik degree of $\{x \mid x$ is recursive$\}$

$\mathbf{1}$ = the top degree in $\mathcal{P}_w$ = the Muchnik degree of $\{x \mid x$ is a completion of Peano Arithmetic$\}$

$\mathbf{r}_1$ = Muchnik degree of $\{x \in \{0,1\}^{\mathbb{N}} \mid x$ is *random*$\}$ (in the sense of Martin-Löf)

$\mathbf{r}_2$ = Muchnik degree of $\{x \in \{0,1\}^{\mathbb{N}} \mid x$ is random relative to $0'$, the Halting Problem$\}$

$\mathbf{d}$ = the Muchnik degree of $\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is *diagonally nonrecursive*$\}$
(i.e., $f(n) \not\simeq \varphi_n^{(1)}(n)$ for all n)

$\mathbf{d}_{\mathsf{REC}}$ = the Muchnik degree of $\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is diagonally nonrecursive and *recursively bounded*$\}$
(i.e., f is bounded by a recursive function)

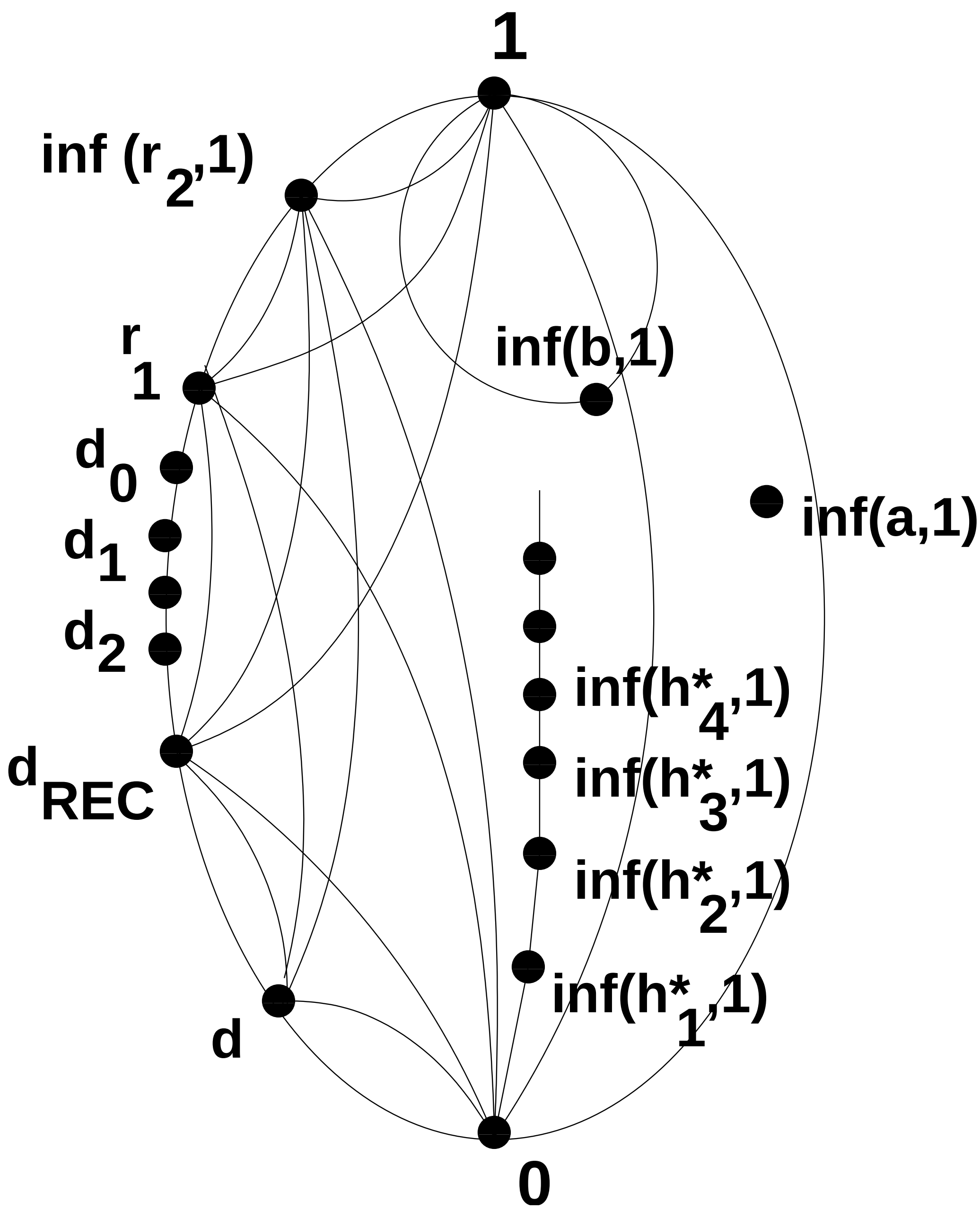
$\mathbf{d}_\alpha$ = the Muchnik degree of $\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is diagonally nonrecursive and α-*recursively bounded*$\}$
(bounded at level α of the Wainer hierarchy), $\alpha \le \varepsilon_0$

$\mathbf{a}$ = the Muchnik degree of a recursively enumerable set

$\mathbf{h}_\alpha$ = the Muchnik degree of $0^{(\alpha)}$, $\alpha < \omega_1^{\mathsf{CK}}$

$\mathbf{h}_\alpha^*$ = the blurred version of $\mathbf{h}_\alpha$, $\alpha < \omega_1^{\mathsf{CK}}$

$\mathbf{b}$ = the Muchnik degree of $\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is almost everywhere dominating$\}$



A picture of $\mathcal{P}_w$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{h} =$ hyperarithmeticity, $\mathbf{r} =$ randomness, $\mathbf{b} =$ almost everywhere domination, $\mathbf{d} =$ diagonal nonrecursiveness.

By Theorem 1, each of the black dots in the above picture is the Muchnik degree of a 2-dimensional subshift of finite type.

Thus we have apparently uncovered some interesting classes of 2-dimensional subshifts of finite type.

A basic result concerning $\mathcal{P}_w$ is as follows:

Embedding Lemma (Simpson 2004). Let $\mathbf{s}$ be the Muchnik degree of a Σ^0_3 set. Then $\inf(\mathbf{s}, \mathbf{1})$ belongs to $\mathcal{P}_w$.

Combining this with Theorem 1, we obtain:

Theorem 4 (Simpson 2007). Let $\mathbf{s}$ be the Muchnik degree of a Σ^0_3 set. Then there exists a 2-dimensional subshift of finite type whose Muchnik degree is $\inf(\mathbf{s}, \mathbf{1})$.

Part 2: INTUITIONISM

Let L be a distributive lattice with 0 and 1.

For $a, b \in L$ let $a \Rightarrow b$ be the unique minimum $x \in L$ such that $\sup(a, x) \geq b$, if it exists.

L is *Brouwerian* if $a \Rightarrow b$ exists for all $a, b \in L$.

Fact: Each Brouwerian lattice is a model of intuitionistic propositional calculus:

$a \wedge b = \sup(a, b)$, $a \vee b = \inf(a, b)$,

$a \Rightarrow b$ as above, $\neg a = (a \Rightarrow 1)$,

$a \vdash b$ if and only if $a \geq b$.

Completeness Theorem (Tarski 1938):
A first-order propositional formula is intuitionistically valid if and only if it is identically 0 in all Brouwerian lattices.

Medvedev 1955: the lattice of all Medvedev degrees is Brouwerian.

Muchnik 1963: the lattice of all Muchnik degrees is Brouwerian.

Historically, mass problems originated from intuitionistic considerations of this kind.

Kolmogorov 1932 informally proposed to view intuitionism as a "*calculus of problems*". This is the famous Brouwer/Heyting/Kolmogorov or BHK interpretation of intuitionism.
See Troelstra/van Dalen, §§ 1.3.1, 1.5.3.

Medvedev 1955 (Kolmogorov's student) introduced Medvedev degrees as a rigorous elaboration of Kolmogorov's proposal.
He noted that the lattice of all Medvedev degrees is Brouwerian.

Muchnik 1963 introduced Muchnik degrees as an alternative rigorous elaboration of Kolmogorov's proposal. He noted that the lattice of all Muchnik degrees is Brouwerian.

Skvortsova 1988.

Sorbi 1990s.

Terwijn 2005 onward.

Starting in 1999 Simpson and others studied the lattices $\mathcal{P}_s$ (respectively $\mathcal{P}_w$) of Medvedev (respectively Muchnik) degrees of nonempty Π^0_1 subsets of 2^ω.

The study of $\mathcal{P}_s$ and especially $\mathcal{P}_w$ has been fruitful, with many specific, natural examples related to foundationally interesting topics.

Also, $\mathcal{P}_s$ and $\mathcal{P}_w$ are distributive with 0 and 1.

Therefore, the question arises:

Are $\mathcal{P}_s$ and $\mathcal{P}_w$ Brouwerian?

Theorem 5 (Simpson 2007).

$\mathcal{P}_w$ is not Brouwerian.

It remains open whether $\mathcal{P}_s$ is Brouwerian.

Terwijn 2005 proved that the dual of $\mathcal{P}_s$ is not Brouwerian.

It remains open whether the dual of $\mathcal{P}_w$ is Brouwerian.

We sketch the proof of Theorem 5.

Let $\leq_T$ denote Turing reducibility.

Let $\leq_w$ denote Muchnik reducibility.

Lemma 1. Assume Q is Π^0_1 and
$Q \nleq_w \{f\}$ and $0 <_T f <_T 0'$.
Then, we can find g such that
$Q \nleq_w \{g\}$ and $0 <_T g <_T 0'$ and $f \oplus g \equiv_T 0'$.

Proof. The special case $Q = \emptyset$ is due to Posner/Robinson 1981. We adapt the technique of that paper. . . . .

To prove Theorem 5, let f and Q be as in Lemma 1 with $Q \neq \emptyset$.
Let $\mathbf{a} = \deg_T(f)$ and $\mathbf{q} = \deg_w(Q)$.
By the Embedding Lemma of Simpson 2004 plus Lemma 1, we have $\inf(\mathbf{a}, \mathbf{q}) \in \mathcal{P}_w$ and $\mathcal{P}_w \models \neg \exists (\inf(\mathbf{a}, \mathbf{q}) \Rightarrow \mathbf{q})$.
Thus $\mathcal{P}_w$ is not Brouwerian.

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