Recent Aspects of Mass Problems: Symbolic Dynamics and Intuitionism

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Mass Problems

A set $P \subseteq \{0,1\}^{\mathbb{N}}$ may be viewed as a *mass problem*, i.e., a decision problem with more than one solution. By definition, the *solutions* of P are the elements of P. A mass problem is said to be *solvable* if at least one of its solutions is recursive. A mass problem P is said to be *weakly reducible* to a mass problem Q if for each solution of Q there exists a solution of P which is Turing reducible to the given solution of Q. A *weak degree* is an equivalence class of mass problems under mutual weak reducibility. The lattice $\mathcal{D}_w$ of all weak degrees is due to Muchnik 1963. There is an obvious embedding of the Turing degrees into $\mathcal{D}_w$.

A set $P \subseteq \{0,1\}^{\mathbb{N}}$ is said to be Π^0_1 if it is *effectively closed*, i.e., it is the complement of the union of a recursive sequence of basic open sets. Let $\mathcal{P}_w$ denote the sublattice of $\mathcal{D}_w$ consisting of the mass problems associated with nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$. The lattice $\mathcal{P}_w$ has been investigated by Simpson and his collaborators. There is a non-obvious but natural embedding of the recursively enumerable Turing degrees into $\mathcal{P}_w$. It is known that $\mathcal{P}_w$ contains many specific, natural weak degrees which are related to various topics in the foundations of mathematics. Among these topics are reverse mathematics, algorithmic randomness, Kolmogorov complexity, almost everywhere domination, hyperarithmeticity, effective Hausdorff dimension, resource-bounded computational complexity, and subrecursive hierarchies.

Symbolic Dynamics

Let A be a finite set of symbols. The *full two-dimensional shift* on A is the dynamical system consisting of the natural action of the group $\mathbb{Z} \times \mathbb{Z}$ on the compact space $A^{\mathbb{Z}\times\mathbb{Z}}$. A *two-dimensional subshift* is a nonempty closed subset of $A^{\mathbb{Z}\times\mathbb{Z}}$ which is invariant under the action of $\mathbb{Z} \times \mathbb{Z}$. A two-dimensional subshift is said to be *of finite type* if it is defined by a finite set of excluded configurations.

The two-dimensional subshifts of finite type are known to form an important class of dynamical systems, with connections to mathematical physics, etc.

Clearly every two-dimensional subshift of finite type is a nonempty Π^0_1 subset of $A^{\mathbb{Z}\times\mathbb{Z}}$, hence its weak degree belongs to $\mathcal{P}_w$. Conversely, we prove that every weak degree in $\mathcal{P}_w$ is the weak degree of a two-dimensional subshift of finite type. The proof of this result uses tilings of the plane. We present an application of this result to symbolic dynamics. Namely, we obtain an infinite family of two-dimensional subshifts of finite type which are, in a certain sense, mutually incompatible. Our application is stated purely in terms of two-dimensional subshifts of finite type, with no mention of weak degrees.

Intuitionism

Historically, the study of mass problems originated from intuitionistic considerations. Kolmogorov 1932 proposed to view intuitionism as a "calculus of problems." Muchnik 1963 introduced weak degrees as a rigorous elaboration of Kolmogorov's proposal. As noted by Muchnik, the lattice $\mathcal{D}_w$ of all weak degrees is Brouwerian.

The question arises, is the sublattice $\mathcal{P}_w$ Brouwerian? We prove that it is not. The proof uses our adaptation of a technique of Posner and Robinson.

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