Medvedev Degrees, Muchnik Degrees, Subsystems of Z_2, and Reverse Mathematics

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Foundations of mathematics (f.o.m.) is the study of the most basic concepts and logical structure of mathematics as a whole. An important f.o.m. research program is reverse mathematics, where one discovers which subsystems of second order arithmetic are necessary and sufficient to prove specific theorems in core mathematical areas: analysis, algebra, geometry, countable combinatorics. One of the most important subsystems for reverse mathematics is WKL_0, consisting of RCA_0 plus Weak König's Lemma.

Let $\mathcal{P}$ be the set of nonempty Π^0_1 subsets of 2^ω. Forcing with $\mathcal{P}$ is known as Jockusch/Soare forcing. I have used iterated Jockusch/Soare forcing to obtain an ω-model M of WKL_0 with the following property: for all $X, Y \in M$, X is definable over M from Y if and only if $X \leq_T Y$. The proof is based on a homogeneity argument involving the Recursion Theorem, and a factorization lemma. I discuss the foundational significance of M and its hyperarithmetical analog.

For $P, Q \in \mathcal{P}$ one says that P is Muchnik reducible to Q if for all $Y \in Q$ there exists $X \in P$ such that $X \leq_T Y$. One says that P is Medvedev reducible to Q if there exists a recursive functional $\Phi : Q \to P$. I introduce the countable distributive lattices $\mathcal{P}_w$ ($\mathcal{P}_M$) consisting of the Muchnik (Medvedev) degrees of members of $\mathcal{P}$. I have shown that $P \in \mathcal{P}$ is Muchnik (Medvedev) complete if and only if P is degree isomorphic (recursively homeomorphic) to the set of complete extensions of PA.

Structural aspects of $\mathcal{P}_w$ and $\mathcal{P}_M$ present a rich problem area for recursion theorists. Stephen Binns and I have shown that every countable distributive lattice is lattice-embeddable below any nonzero degree in $\mathcal{P}_w$. We have also obtained partial results in this direction for $\mathcal{P}_M$.

The lattices $\mathcal{P}_w$ and $\mathcal{P}_M$ are in some respects similar to the upper semilattice of Turing degrees of recursively enumerable subsets of ω. However, $\mathcal{P}_w$ and $\mathcal{P}_M$ are much better in that they contain specific, known, natural examples of degrees $\neq 0, 1$. Such examples are especially relevant for f.o.m. In $\mathcal{P}_w$ there is the maximum Muchnik degree of Π^0_1 subsets of 2^ω of positive measure. This is of interest for the reverse mathematics of measure theory. In $\mathcal{P}_M$ there are the Medvedev degrees of $\{X : X \text{ is } k\text{-valued DNR}\}$, $k \geq 3$. This is of interest for the reverse mathematics of graph coloring.

For references see http://www.math.psu.edu/simpson/talks/obwf0101/.