Medvedev and Muchnik Degrees of Nonempty $\mathbf{\Pi^0_1}$ Subsets of $\mathbf{2}^\omega$

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We use 2^ω to denote the space of infinite sequences of 0's and 1's. For $X, Y \in 2^\omega$, $X \le_T Y$ means that X is Turing reducible to Y. For $P, Q \subseteq 2^\omega$ we say that *P is Muchnik reducible to Q*, abbreviated $P \le_w Q$, if for all $Y \in Q$ there exists $X \in P$ such that $X \le_T Y$. We say that *P is Medvedev reducible to Q*, abbreviated $P \le_M Q$, if there exists a recursive functional $\Phi : Q \to P$. Note that $P \le_M Q$ implies $P \le_w Q$, but not conversely.

Sorbi [13] gives a useful survey of Medvedev and Muchnik degrees of arbitrary subsets of 2^ω. Here we initiate a study of Medvedev and Muchnik degrees of nonempty Π^0_1 subsets of 2^ω. Theorems 1 and 2 below have been proved in collaboration with my Ph. D. student Stephen Binns.

Let $\mathcal{P}$ be the set of nonempty Π^0_1 subsets of 2^ω. Let $\mathcal{P}_M$ (respectively $\mathcal{P}_w$) be the set of Medvedev (respectively Muchnik) degrees of members of $\mathcal{P}$, ordered by Medvedev (respectively Muchnik) reducibility. We say that $P \in \mathcal{P}$ is *Medvedev complete* (respectively *Muchnik complete*) if $P \ge_M Q$ (respectively $P \ge_w Q$) for all $Q \in \mathcal{P}$. For example, the set of complete extensions of Peano Arithmetic is Medvedev complete. Clearly every Medvedev complete set is Muchnik complete, but not conversely. $\mathcal{P}_M$ and $\mathcal{P}_w$ are countable distributive lattices with a bottom element, $\mathbf{0}$, the degree of 2^ω, and a top element, $\mathbf{1}$, the degree of any Medvedev complete set. The lattice operations are given by $P \times Q = \{X \oplus Y : X \in P \text{ and } Y \in Q\}$ and $P + Q = \{\langle 0\rangle^\frown X : X \in P\} \cup \{\langle 1\rangle^\frown Y : Y \in Q\}$. Here $P \times Q$ (respectively $P + Q$) is the join (respectively meet) of P and Q.

In both $\mathcal{P}_M$ and $\mathcal{P}_w$, it is easy to see that $P, Q > \mathbf{0}$ implies $P + Q > \mathbf{0}$, but we do not know whether $P, Q < \mathbf{1}$ implies $P \times Q < \mathbf{1}$. In addition, there are many other open problems. For example, we do not know whether the Sacks Density Theorem holds for $\mathcal{P}_M$ or for $\mathcal{P}_w$. This subject appears to be a rich source of problems for recursion theorists.

Theorem 1 *In $\mathcal{P}_w$, for every $P > \mathbf{0}$, every countable distributive lattice is lattice embeddable below P.*

Proof (sketch). We begin by defining infinitary "almost lattice" operations in such a way that, given a recursive sequence $\langle P_i : i \in \omega\rangle$ of members of $\mathcal{P}$, $\prod_{i\in\omega} P_i$ and $\sum_{i\in\omega} P_i$ are again members of $\mathcal{P}$. We define the infinite product in the obvious way, $\prod_{i\in\omega} P_i = \{X \in 2^\omega : (X)_i \in P_i \text{ for all } i \in \omega\}$, where $(X)_i(n) = X((n,i))$ for all $n \in \omega$. To define the infinite sum, let $\langle T_i : i \in \omega\rangle$ be a recursive sequence of recursive subtrees of $2^{<\omega}$ such that, for each i, P_i is the set of paths through T_i. Let R be a fixed, Medvedev complete member of $\mathcal{P}$. Let T be a recursive subtree of $2^{<\omega}$ such that R is the set of paths through T. Put $\widetilde{T} = \{\tau^\frown\langle k\rangle : \tau \in T, k \in \{0,1\}, \tau^\frown\langle k\rangle \notin T\}$. Let $\langle \sigma_i : i \in \omega\rangle$ be a recursive enumeration of $\widetilde{T}$ without repetition. We define the infinite sum $\sum_{i\in\omega} P_i$ to be the set of paths through $T^* = T \cup \{\sigma_i^\frown\tau : i \in \omega, \tau \in T_i\}$.

To prove the theorem, construct a recursive sequence $\langle Q_i : i \in \omega\rangle$ of members of $\mathcal{P}$ such that $X \not\leq_T Y$ for all $X \in Q_i$, $Y \in Q_j$, $i \neq j$. This is a finite injury priority construction, similar to that of Jockusch/Soare [6, Theorem 4.7]. For $i \in \omega$ define $\hat{Q}_i = \sum_{j\neq i} Q_j$. For recursive $A \subseteq \omega$ define $\hat{Q}(A) = \prod_{i\in A} \hat{Q}_i$. We have $\hat{Q}(A) \in \mathcal{P}$ and $\hat{Q}(A \cup B) \equiv_w \hat{Q}(A) \times \hat{Q}(B)$ and $\hat{Q}(A \cap B) \equiv_w \hat{Q}(A) + \hat{Q}(B)$. Moreover $A \neq B$ implies $\hat{Q}(A) \not\equiv_w \hat{Q}(B)$. Thus $A \mapsto \hat{Q}(A)$ is a lattice embedding of the lattice of recursive subsets of ω into $\mathcal{P}_w$. Note also that every countable distributive lattice is lattice embeddable into the lattice of recursive subsets of ω. To push the embedding below P, assume in addition that $X \not\leq_T Y$ for all $X \in P$, $Y \in Q_i$, $i \in \omega$. This property can be obtained by a Sacks preservation strategy. Our lattice embedding below P is given by $A \mapsto \hat{Q}(A) + P$. $\square$

It seems likely that Theorem 1 holds with $\mathcal{P}_w$ replaced by $\mathcal{P}_M$. In this direction we have the following partial result.

Theorem 2 *In $\mathcal{P}_M$, for every $P > \mathbf{0}$, the free countable distributive lattice is lattice embeddable below P, as are the lattice of finite subsets of ω, the lattice of cofinite subsets of ω, and all finite distributive lattices.*

Proof (sketch). With notation as before, we have $\hat{Q}(A \cup B) \equiv_M \hat{Q}(A) \times \hat{Q}(B)$ and $\hat{Q}(A \cap B) \equiv_M \hat{Q}(A) + \hat{Q}(B)$, provided $A, B \subseteq \omega$ are finite. Thus $A \mapsto \hat{Q}(A) + P$ is the desired lattice embedding of the lattice of finite subsets of ω into $\mathcal{P}_M$ below P. Note also that every finite distributive lattice is lattice embeddable into the lattice of finite subsets of ω.

Now assume that $\langle Q_i : i \in \omega \rangle$ has the following stronger properties: (1) $X_i \not\leq_T Y$ for all $X_i \in Q_i$, $Y \in \prod_{j \neq i} Q_j$, (2) $X \not\leq_T Y$ for all $X \in P$, $Y \in \prod_{i \in \omega} Q_i$. As before, these properties can be obtained by a finite injury priority argument and a preservation strategy. For $i \in \omega$ define $\check{Q}_i = \prod_{j \neq i} Q_i$. For recursive $A \subseteq \omega$ define $\check{Q}(A) = \sum_{i \in A} \check{Q}_i$. Note that the definition of $\check{Q}(A)$ is dual to that of $\hat{Q}(A)$. We have $\check{Q}(A) \in \mathcal{P}$ and $\check{Q}(A \cap B) \equiv_w \check{Q}(A) \times \check{Q}(B)$ and $\check{Q}(A \cup B) \equiv_w \check{Q}(A) + \check{Q}(B)$. For finite $A, B \subseteq \omega$ we have $\check{Q}(A \cap B) \equiv_M \check{Q}(A) \times \check{Q}(B)$ and $\check{Q}(A \cup B) \equiv_M \check{Q}(A) + \check{Q}(B)$. Thus $\omega \setminus A \mapsto \check{Q}(A) + P$ is the desired lattice embedding of the lattice of cofinite subsets of ω into $\mathcal{P}_M$ below P.

Finally, with Q_i, $i \in \omega$ as above, the Medvedev degrees of $Q_i + P$, $i \in \omega$ freely generate a free distributive sublattice of $\mathcal{P}_M$ below P. $\square$

Remark The study of the distributive lattices $\mathcal{P}_M$ and $\mathcal{P}_w$ is in some ways parallel to the study of $\mathcal{R}_T$, the upper semilattice of Turing degrees of recursively enumerable subsets of ω. (For background on this topic, see Soare [12].) However, as is well known, there are no specific examples of recursively enumerable degrees $\neq \mathbf{0}, \mathbf{0}'$. (See the FOM discussion with Soare [4, July 1999].) In this respect, $\mathcal{P}_M$ and $\mathcal{P}_w$ are much better, as shown by the following two theorems, due to Kučera [7] and Jockusch [5] respectively.

Theorem 3 *Among all Muchnik degrees of Π^0_1 subsets of 2^ω of positive measure, there is a unique largest one. This particular element of $\mathcal{P}_w$ is $\neq \mathbf{0}, \mathbf{1}$.*

Theorem 4 *For $k \geq 2$ let* DNR_k *be the Π^0_1 set of k-valued* DNR *functions. In the Medvedev degrees $\mathcal{P}_M$ we have $\mathbf{1} \equiv_M \mathrm{DNR}_2 >_M \mathrm{DNR}_3 >_M \cdots >_M \mathrm{DNR}_k >_M \cdots >_M \mathbf{0}$. All of these Medvedev degrees are Muchnik complete.*

Simpson [11, Theorem 3.20] proves the following analog of Myhill's Theorem on creative sets (see Rogers [9]). This is closely related to a result of Pour-El/Kripke [8].

Theorem 5 *If $P, Q \in \mathcal{P}$ are Medvedev complete, then P and Q are recursively homeomorphic.*

Proof (sketch). First we define what it means for $P \in \mathcal{P}$ to be *productive*. Then we use the Recursion Theorem to prove the following two results: (1) P is productive if and only if P is Medvedev complete. (2) Any two productive sets $P, Q \in \mathcal{P}$ are recursively homeomorphic. The details are in Simpson [11, Section 3]. $\square$

Simpson [11, Theorem 6.9] applies Theorem 5 plus Jockusch/Soare forcing [6, Theorem 2.4] to prove the following result concerning subsystems of second order arithmetic. (For background on this topic, see Simpson [10].)

Theorem 6 *There is a countable ω-model S of WKL_0 with the following property. For all $X, Y \in S$, X is definable from Y over S if and only if $X \leq_T Y$.*

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