Toward objectivity in mathematics

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Abstract

We present some ideas in furtherance of objectivity in mathematics. We call for closer integration of mathematics with the rest of human knowledge. We note some insights which can be drawn from current research programs in the foundations of mathematics.

Contents

This is the text of my talk at a conference on philosophy of mathematics at New York University, April 3–5, 2009. I wish to thank Justin Clarke-Doane and Shieva Kleinschmidt for their attention to detail in organizing this conference. Moreover, I am excited about this opportunity to address a wonderful audience at a great urban university in the greatest city in the world. As the saying goes, it doesn't get any better than this!

1 Objectivity and Objectivism

I am a mathematician, not a philosopher. However, as a mathematician and a human being, I have always had the greatest respect for philosophy, and I have always recognized the need for philosophical guidance.

My thinking is largely informed by a particular philosophical system:

Objectivism (with a capital "O").

A key reference for me is Leonard Peikoff's treatise [4]. By the way, Peikoff obtained his Ph.D. degree in Philosophy here at New York University in 1964. His thesis advisor was Sidney Hook.

For those not familiar with Objectivism, let me say that it is a coherent, integrated, philosophical system which encompasses the five main branches of philosophy: metaphysics, epistemology, ethics, politics, aesthetics.

As the name "Objectivism" implies, the concept of *objectivity* plays a central role in the system. Because objectivity is an epistemological concept, let me say a little about the Objectivist epistemology. Of course, my brief account of the Objectivist epistemology cannot be fully understood outside the context of certain other aspects of Objectivism which I do not plan to discuss here.

The main point is that Objectivist epistemology calls for a close relationship between *existence* (the reality which is "out there") and *consciousness* (a volitional process that takes place within the human mind).

1. According to Objectivism, *knowledge* (i.e., human, conceptual knowledge) is "grasp of an object by means of an active, reality-based process which is chosen by the subject."

2. According to Objectivism, *objectivity* is a specific kind of relationship between reality ("out there") and consciousness ("in here").

3. All knowledge is *contextual*, i.e., it must be understood within a context. Moreover, the ultimate context is the totality of human knowledge. Thererore, all of human knowledge must be integrated into a coherent system. Compartmentalization is strongly discouraged (more about this later).

4. In integrating human knowledge into a coherent whole, the method of integration is *logic*, defined as "the art of non-contradictory identification." Here "identification" refers to the conceptual grasp of an object or entity in reality.

5. All knowledge is *hierarchical*. Concepts must be justified or validated by reference to earlier concepts, which are based on still earlier concepts, etc., all the way down to the perceptual roots. This validation process is called *reduction*.

We may contrast Objectivism with two other types of philosophy: *intrinsicism* (e.g., Plato, Augustine) and *subjectivism* (e.g., Kant, Dewey).

1. To their credit, the *intrinsicists* recognize that knowledge must conform to reality. However, intrinsicism goes overboard by denying the active or volitional nature of consciousness. According to intrinsicism, the process of acquiring knowlege is essentially passive. It consists of "revelation" (Judeo-Christian theology) or "remembering" (Plato) or "intuition," not volitional activity. The operative factor is *existence* rather than consciousness.

2. To their credit, the *subjectivists* recognize that revelation is not a valid means of cognition. However, subjectivism goes too far by insisting that concepts are not based on reality but rather are created solely out of the resources of our own minds. There are several versions of subjectivism. In the personal version, each individual creates his own universe. In the social or collective version, concepts and facts are created by a group. In all versions of subjectivism, the operative factor is *consciousness* rather than existence.

Objectivism strikes a balance by emphasizing a close relationship between existence and consciousness. Each of these two factors is operative. Their close relationship is summarized in a slogan:

> "Existence is identity; consciousness is identification."

2 Mathematics as part of human knowledge

A major problem in universities and in society generally is *compartmentalization.* Compartmentalization is a kind of overspecialization in which one regards one's own specialty as an isolated subject, unrelated to the rest of human knowledge. Thus, the teachings of one university department (e.g., the English department) may flatly contradict those of another (e.g., the business school) and this kind of situation is regarded as normal.

Compartmentalization can sometimes exist within a single individual. An example is the conservative economist who advocates the profit motive in economics and the Sermon on the Mount in church. Another example is the legislator who calls for strict government control of political advocacy and commercial activity, while at the same time paying lip service to freedom of speech and association.

Here I wish to focus on compartmentalization in the university context, with which I am very familiar.

I am a professor of mathematics at a large state university, Penn State. At our main campus in the appropriately named Happy Valley in Pennsylvania, there are more than 40,000 undergraduate students as well as thousands of graduate students and postdocs.

At the Pennsylvania State University as at most other large universities, much of the research activity is mathematical in nature. Mathematics, statistics, and large-scale computer simulations are heavily used as research tools. This applies to the majority of academic divisions of the university: not only physical sciences and engineering, but also biological sciences, agriculture, business, social sciences, earth sciences, materials science, medicine, and even humanities. In addition Penn State has an Applied Research Laboratory which performs classified, defense-related research and has a huge annual budget. There also, mathematics is heavily used.

What is interesting is that our Department of Mathematics is largely uninvolved in this kind of activity. When mathematicians and non-mathematicians try to collaborate, both sides are often frustrated by "communication difficulties" or "failure to find common ground," due largely to lack of a common vocabulary and conceptual framework. To me this widespread frustration suggests a failure of integration.

As an aside, we can see the detrimental effects of a lack of mathematics in public affairs. Basic mathematical and statistical knowledge is astonishingly rare among the voting public. Lack of quantitative understanding of relative benefits and relative risks may stifle innovation. “A trillion is the new billion,” and angry mobs with pitchforks may lose sight of the decimal point.

But, back to the university context. From interactions with mathematicians and non-mathematicians at Penn State and elsewhere, I see a need for greater integration of mathematics with the rest of human knowledge. We need to somehow overcome the compartmentalization which isolates mathematics from application areas.

Philosophy is the branch of knowledge that deals with the widest possible abstractions – concepts such as justice, friendship, and objectivity. Therefore, only philosophy can act as the ultimate integrator of human knowledge. A crucial task for philosophers of mathematics is to provide general principles which can guide both mathematicians and users of mathematics.

Some of the most pressing issues involve *mathematical modeling*. By a mathematical model I mean an abstract mathematical structure S (e.g., a system of differential equations) together with a claimed relationship between S and a real-world situation R (e.g., a weather system). Typically, the mathematician designs the structure S, and the non-mathematician decides which assumptions (e.g., initial conditions) are to be fed into S and how to interpret the results in R. Such models are used extensively in engineering, finance, economics, climate studies, etc.

Some currently relevant questions about mathematical modeling are as follows. What are the appropriate uses of quantitative financial models in terms of risk and reward? Would it be ethical to incorporate the prospect of government bailouts into such models? What are appropriate limitations on the role of mathematical modeling in climate studies? Under what circumstances is it ethically appropriate to base public policy on such models? Etc., etc.

By nature such questions are highly interdisciplinary and require a broad perspective. Therefore, it seems reasonable to think that such questions may be a proper object of study for philosophers of mathematics. It would be wonderful if philosophers could provide a valid framework or standard for answering such questions. Of course, it goes without saying that this kind of philosophical activity would have to be based on a coherent philosophical system including an integrated view of human knowledge as a whole and the role of mathematics within it.

3 Set theory and the unity of mathematics

As is well known, mathematicians tend to group themselves into research specialties: analysis, algebra, number theory, geometry, topology, combinatorics, ordinary differential equations, partial differential equations, mathematical logic, etc. Each of these groups holds its own conferences, edits its own journals, writes letters of recommendation for its own members, etc. Furthermore, among these groups there is frequent and occasionally bitter rivalry with respect to academic hiring, research professorships, awards, etc.

As an antidote to this kind of fragmentation, high-level mathematicians frequently express an interest in promoting *the unity of mathematics*. An 11th commandment for mathematicians has been proposed:

> "Thou shalt not criticize any branch of mathematics."

A variant reads as follows:

> "*All* mathematics is difficult; *all* mathematics is interesting."

Partly as a result of such considerations, research programs which combine several branches of mathematics are highly valued. Examples of such programs are algebraic topology, geometric functional analysis, algebraic geometry, geometric group theory, etc. Such programs are regarded as valuable partly because they draw together two or more research sub-communities within the larger mathematical community.

As regards the unity of mathematics, set theory has made at least one crucial contribution. Namely, the well known formalism of ZFC, Zermelo-Fraenkel set theory (based on classical first-order logic and including the Axiom of Choice) is a huge achievement. The ZFC formalism provides two extremely important benefits for mathematics as a whole: a common framework, and a common standard of rigor.

1. ZFC provides the orthodox, commonly accepted framework for virtually all of contemporary mathematics. Indeed, advanced undergraduate textbooks in almost all branches of mathematics frequently include either an appendix or an introductory chapter outlining the common set-theoretic notions: sets, functions, union, intersection, Cartesian product, etc.

2. The ZFC framework is sufficiently simple and elegant so that all mathematicians can easily gain a working knowledge of it. There is only

one basic concept: *sets*. The axioms of ZFC consist of easily understood, plausible, self-evident assumptions concerning the universe of sets. Moreover, the ZFC framework is flexible and far-reaching; within it one can easily and quickly construct isomorphs of all familiar mathematical structures including the natural number system, the real number system, Euclidean spaces, manifolds, topological spaces, Hilbert space, operator algebras, etc.

3. Among mathematicians, there is little or no controversy about what it would mean to rigorously prove a mathematical theorem. All such questions are answered by saying that the proof must be formalizable in ZFC, i.e., deducible from the axioms of ZFC using standard logical axioms and rules. In his talk yesterday, Professor Gaifman gave an admirably detailed description of how this ZFC-based verification process works in practice.

   It is noteworthy that similarly clear standards of rigor do not currently exist in other sciences such as physics, economics, or philosophy.

4. Mathematicians are highly appreciative of the existence of a common framework and standard of rigor such as ZFC provides.

   For instance, there is currently little or no controversy surrounding the Axiom of Choice such as took place in the early 20th century. Virtually all mathematicians are happy and relieved to know that this and similar controversies have been laid to rest.

5. This comfortable situation allows "working mathematicians" to get on with their research, secure in the belief that they will not be undercut by some obscure foundational brouhaha. Mathematicians appreciate ZFC because it seems to relieve them of the need to bother with foundational questions.

On the other hand, mathematicians have some justifiable reservations about set-theoretic foundations. The existence of a variety of models of ZFC (the set-theoretic "multiverse") is somewhat unsettling, at least for those mathematicians who take foundations seriously. Some mathematicians deal with this kind of uncertainty by asserting that questions such as the Continuum Hypothesis and large cardinals are unlikely to impinge on their own branch of mathematics, or at least their own research within that branch.

Some mathematicians even make a point of avoiding higher set theory, for fear of running into such scary monsters. (Of course we mathematical logicians know or strongly suspect that they are whistling in the dark.)

Even worse, when we contemplate the philosophical task which was outlined in Section 2 above, the program of set-theoretic foundations based on systems such as ZFC seems unhelpful to say the least. There seems to be no clear path toward integration of set theory with the rest of human knowledge. Infinite sets and the cumulative hierarchy present a stumbling block. It is completely unclear how to reduce a concept such as $\aleph_\omega$ to referents in "the real world out there." We have no idea whatsoever of how to understand the Continuum Hypothesis as a question about "the real world out there." What in "the real world out there" are the set theorists talking about? The answer seems unclear, and nobody can agree on how to proceed.

Thus it emerges that the program of set-theoretic foundations, useful though it has been in promoting the unity of mathematics and defining a standard of mathematical rigor, appears to stand as an obstacle in the way of a highly desirable unification of mathematics with the rest of human knowledge.

Indeed, by encouraging the mathematical community to live in relative complacency with respect to foundational issues, the program of set-theoretic foundations may actually be leading us away from fundamental tasks which are clearly of great philosophical importance. The unity of mathematics is valuable, but the unity of human knowledge would be much more valuable.

4 Set-theoretic realism

4.1 An epistemological question

Some high-level set theorists such as Gödel, Martin, Steel, and Woodin, as well as some high-level philosophers of mathematics such as Maddy, have advocated a philosophical position known as *set-theoretic realism* or Platonism. According to this program, set theory refers to certain definite, undeniable aspects of reality. For instance, cardinals such as $\aleph_\omega$ are thought to exist in a certain domain of reality, and the Continuum Hypothesis is thought to be a meaningful statement about that domain.

An epistemological question which remains is:

> How can we acquire knowledge of the set-theoretic reality?

We briefly consider three contemporary answers to this question.

4.2 The intrinsicist answer

One answer is that the set-theoretic reality is a non-spatial, non-temporal, irreducible kind of reality which reveals itself by means of pure intuition. I have no response to this, except to say that it seems to express an intrinsicist viewpoint which is obviously incompatible with the requirement of objectivity as I understand it.

4.3 The "testable consequences" answer

Another answer to our epistemological question says that the higher set-theoretic reality, although not directly observable, may reveal itself by means of "testable" logical consequences in the concrete mathematical realm. For instance, by Matiyasevich's Theorem, the consistency of a large cardinal axiom can be recast as a number-theoretic statement to the effect that a certain Diophantine equation has no solution in the integers. The resulting justification process for large cardinals is said to be analogous to how the atomic theory of matter was originally discovered and verified, long before it became possible to observe individual atoms directly under an electron microscope.

I find this "testable consequences" viewpoint more appealing than the purely intrinsicist viewpoint, because it gives an active role to a human cognitive process, namely, the study of concrete mathematical problems such as Diophantine equations. Higher set theory is to be justified or reduced or "miniaturized" in terms of its applications to down-to-earth mathematical problems.

The major difficulty that I see with the "testable consequences" program involves its implemention. For instance, the Diophantine equations which have been produced in the manner outlined above are messy and complex and have thousands of terms. No number theorist would seriously study such an equation. Thus, the value of such equations for number theory seems remarkably tenuous. By contrast, the atomic theory from its inception produced a powerful stream of striking consequences in chemistry and other fields of knowledge. These consequences greatly improved the human standard of living.

Attempting to overcome the implementation difficulty, set-theorists have worked very hard for many years trying to uncover consequences of higher set theory and large cardinals which are not only down-to-earth but also *mathematically appealing* and perhaps even *useful in applications*. I am thinking of the impressive results of Martin, Steel and Woodin [2] on projective determinacy,[1] and of Harvey Friedman (unpublished) on Boolean relation theory.

And yet, appealing as they may be, these consequences of large cardinal axioms remain quite remote from standard mathematical practice, especially in application areas. Partly for this reason, they have not led to an upsurge of interest in higher set theory and large cardinals within the mathematical community beyond set theory. Indeed, considering all the hard work that has already gone into this research direction, the prospect of serious impact in core mathematics or in mathematical application areas seems even more unlikely than before.

4.4 The Thin Realist answer

Another answer to our epistemological question is Maddy's current philosophy of Thin Realism [3] (in contrast to her earlier Robust Realism, i.e., pure intrinsicism). According to Thin Realism, set theory is in a very strong epistemological position, simply because it is deeply embedded in the "fabric of mathematical fruitfulness." Here again I have my doubts, for the same reasons as above.

Maddy even goes so far as to compare large cardinals to tables and chairs, and set theory skeptics to evil daemon theorists. In other words,

$$\frac{\text{large cardinals}}{\text{set theory skepticism}} = \frac{\text{tables and chairs}}{\text{evil daemon theories}}.$$

Indeed, according to Maddy, our knowledge of set theory is *more* reliable than our knowledge of tables and chairs, because sense perceptions are subject to skeptical doubts which cannot possibly apply to the "fabric of mathematical fruitfulness."

[1] However, Hugh Woodin notes that this research on projective determinacy was motivated not by the "testable consequences" program, but rather by the desire to answer some long-standing structural questions in the branch of mathematics known as *descriptive set theory* (the study of projective sets in Euclidean space, going back to Souslin and Lusin).

My view is that, instead of comparing large cardinals to tables and chairs, it seems more appropriate to compare set theory to religion. In other words,

$$\frac{\text{large cardinals}}{\text{set theory skepticism}} = \frac{\text{gods and devils}}{\text{religious skepticism}}.$$

The point of my analogy is that both set theory and religious faith can claim to be in a "strong" position vis a vis skeptics, to the extent that they avoid dependence on underlying facts of reality which can be questioned. In my view, such claims must be rejected on grounds of their lack of objectivity.

Nevertheless, I applaud Maddy's "Second Philosopher" for her earnest attempt to apply standard scientific or epistemological criteria following the lead of other sciences such as biology. It would be very desirable to flesh this out into a full-scale integration of mathematics with the rest of human knowledge.

5 Insights from reverse mathematics

For many years I have been involved in a foundational research program known as *reverse mathematics*. The purpose of reverse mathematics is to classify core mathematical theorems according to the set existence axioms which are needed to prove them. Frequently it turns out that a core mathematical theorem is logically equivalent to the weakest such set existence axiom. Hence the name "reverse mathematics." The program has revealed an interesting logical structure within core mathematics. In particular, a large number of core mathematical theorems fall into a small number of logical equivalence classes. Moreover, the set existence axioms which arise in this way are naturally arranged in a hierarchy corresponding roughly to Gödel's hierarchy of consistency strengths. The basic reference on reverse mathematics is my book [6]. Table 1 is from my recent paper [7].

I believe that many results of reverse mathematics are potentially useful for answering certain questions and evaluating certain programs in the philosophy of mathematics. As regards objectivity in mathematics, I see two insights to be drawn:

1. A series of reverse mathematics case studies has shown that the bulk of core mathematical theorems falls at the lowest levels of the hierarchy: WKL_0 and below. The full strength of first-order arithmetic appears

$$
\begin{array}{rl}
\text{strong} & \left\{\begin{array}{l}
\vdots \\
\text{supercompact cardinal} \\
\vdots \\
\text{measurable cardinal} \\
\vdots \\
\mathsf{ZFC}\ \text{(Zermelo/Fraenkel set theory)} \\
\mathsf{ZC}\ \text{(Zermelo set theory)} \\
\text{simple type theory}
\end{array}\right. \\
\\
\text{medium} & \left\{\begin{array}{l}
\mathsf{Z}_2\ \text{(second-order arithmetic)} \\
\vdots \\
\Pi^1_2\text{-}\mathsf{CA}_0\ (\Pi^1_2\ \text{comprehension}) \\
\Pi^1_1\text{-}\mathsf{CA}_0\ (\Pi^1_1\ \text{comprehension}) \\
\mathsf{ATR}_0\ \text{(arithmetical transfinite recursion)} \\
\mathsf{ACA}_0\ \text{(arithmetical comprehension)}
\end{array}\right. \\
\\
\text{weak} & \left\{\begin{array}{l}
\mathsf{WKL}_0\ \text{(weak König's lemma)} \\
\mathsf{RCA}_0\ \text{(recursive comprehension)} \\
\mathsf{PRA}\ \text{(primitive recursive arithmetic)} \\
\mathsf{EFA}\ \text{(elementary function arithmetic)} \\
\text{bounded arithmetic} \\
\vdots
\end{array}\right.
\end{array}
$$

Table 1: Some benchmarks in the Gödel hierarchy.

often but not nearly so often as WKL_0. The higher levels up to Π^1_2-CA_0 appear sometimes but rarely. For details see [6] and [7].

To me this strongly suggests that higher set theory is, in a sense, largely irrelevant to core mathematical practice. Thus the program of set-theoretic foundations is once again called into question.

2. It is known that the lowest levels of the Gödel hierarchy (see Table 1) are conservative over PRA (primitive recursive arithmetic) for Π^0_2 sentences. This result combined with reverse mathematics is the basis of some rather strong partial realizations of Hilbert's program of finitistic reductionism, as outlined in my paper [5]. The upshot is that a large portion of core mathematics, sufficient for applications, can be validated by reference to principles which are finitistically provable. It seems to me that these results may open a path toward objectivity in mathematics.

6 Wider cultural significance?

Throughout history we see various trends in the philosophy of mathematics, and we see various trends in the culture at large. Are there parallels here? The intrinsicist/subjectivist dichotomy, to which I alluded earlier, may provide some clues.

Clearly mathematics played a large role in the philosophy of Plato and Aristotle and in the Renaissance, the Enlightment, and the 19th century. However, let us skip ahead to the 20th century.

A thoroughly subjectivistic philosophy of mathematics was Brouwer's Intuitionism. According to Brouwer, mathematics consists of constructions which are performed in the mind of a "creative subject," with no necessary relation to reality. Surely there is a parallel with the subjectivism and collectivism of the early 20th century.

On the intrincist side, consider the rise of religious fundamentalism in the late 20th century: Islamic fundamentalism in the Muslim world, Christian and Jewish fundamentalism in the west, Hindu fundamentalism in India. Could it be that the late 20th century trend toward set-theoretic realism parallels the worldwide rise of religious fundamentalism? This could make an interesting topic of dinner conversation this evening ....

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