Toward objectivity in mathematics

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Abstract: We present some ideas in furtherance of objectivity in mathematics. We call for closer integration of mathematics with the rest of human knowledge. We note some insights which can be drawn from current research programs in the foundations of mathematics.

First, a big thank you to Justin Clarke-Doane and Shieva Kleinschmidt.

Outline of this talk:

1. Objectivity and Objectivism

2. Set theory and the unity of mathematics

3. Mathematics as part of human knowledge

4. Set-theoretic realism

5. Insights from reverse mathematics

6. Wider cultural significance?

1. Objectivity and Objectivism.

Our comments are informed by a particular philosophical system:

Objectivism (with a capital “O”).

Reference: *Objectivism: The Philosophy of Ayn Rand* [4], by Leonard Peikoff (NYU Ph.D. in Philosophy).

Objectivism is an integrated philosophical system. We comment only on the Objectivist epistemology.

The main point is a close relationship between *existence* (“out there”) and *consciousness* (“in here”).

The Objectivist epistemology:

1. *knowledge*: "grasp of an object by means of an active, reality-based process which is chosen by the subject."

2. *objectivity*: a specific relationship between exsitence and consciousness.

3. *context*. All knowledge is contextual and must therefore be integrated into a coherent whole.

4. *compartmentalization*: a failure of integration (more about this later).

5. *logic*: "the art of non-contradictory identification."

6. *hierarchy*. Concepts are validated by reference to earlier concepts, etc., down to the perceptual roots.

To understand Objectivism, contrast it with two other kinds of philosphies:

intrinsicism and subjectivism.

1. *intrinsicism*: acknowledges reality but denies the volitional role of consciousness. Knowledge is acquired by revelation or intuition.

2. *subjectivism*: acknowledges consciousness but denies the role of reality. Knowledge is created by an individual or a group.

Objectivism strikes a balance:

> "Existence is identity;
> consciousness is identification."

2. Mathematics as part of human knowledge.

Compartmentalization in the university environment.

Compartmentalization within individuals.

The Pennsylvania State University.

Lack of integration of mathematics with application areas: physical sciences, earth sciences, social sciences, engineering, etc.

Mathematics in public affairs.

Philosophy is responsible for integrating human knowledge into a coherent whole.

Mathematical modeling.

Some philosophical questions.

3. The unity of mathematics.

Specialties within mathematics:

geometry, number theory, differential equations, mathematical logic, etc.

An antidote: the unity of mathematics.

Combinations of specialties:

algebraic geometry, geometric analysis, etc.

Set theory contributes to the unity of mathematics, by providing a common framework and a common standard of rigor.

Namely, ZFC = Zermelo/Fraenkel set theory, including the Axiom of Choice.

The unity of mathematics, continued.

Set theory and the unity of mathematics.

1. ZFC as a common framework.

2. ZFC as a standard of rigor.

3. ZFC as a comfortable answer to foundational questions.

On the other hand, there are justified qualms:

1. The set-theoretic “multiverse”.

2. Avoidance of higher set theory.

3. There is no clear way to integrate ZFC-based mathematics with the rest of human knowledge.

The unity of mathematics is good, but the unity of human knowledge would be better.

4. Set-theoretic realism.

Gödel, Martin, Steel, Woodin, Maddy.

According to set-theoretic realism, set theory refers to certain aspects of reality.

Examples: $\aleph_\omega$, the Continuum Hypothesis.

A key epistemological question:

> How can we acquire knowledge of the set-theoretic reality?

We consider three contemporary answers.

A. The intrinsicist answer.

B. The “testable consequences” answer.

C. The Thin Realist answer.

Acquiring knowledge of set-theoretic reality.

A. The intrinsicist answer: pure intuition.

This seems incompatible with the requirement of objectivity.

B. Testable consequences.

Example: Diophantine equations.

Analogy with the atomic theory of matter.

The “testable consequences” answer is different from the intrinsicist answer, because it gives an active role to cognitive processes.

Testable consequences, continued.

The difficulty is in the implementation.

E.g., the Diophantine equations are too complicated.

Projective determinacy [2],
Boolean relation theory.

These consequences are remote from core mathematics and application areas.

C. Thin Realism.

This is Maddy's current view, in contrast to her earlier Robust Realism. She argues that set-theoretic realism is embedded in the "fabric of mathematical fruitfulness."

I have my doubts, as above.

Maddy's analogy:

$$\frac{\text{large cardinals}}{\text{set theory skepticism}} = \frac{\text{tables and chairs}}{\text{evil daemon theories}}.$$

I propose a competing analogy:

$$\frac{\text{large cardinals}}{\text{set theory skepticism}} = \frac{\text{gods and devils}}{\text{religious skepticism}}.$$

Thin Realism, continued.

The point of my analogy is that set theory and religious faith can claim to be in a "strong" position vis a vis skeptics, by avoiding reliance on facts which can be questioned.

I reject such claims on grounds of lack of objectivity.

However, I applaud Maddy for attempting to apply standard scientific criteria.

Can this be developed into a full-scale integration of mathematics with the rest of human knowledge?

5. Insights from reverse mathematics.

Reverse mathematics classifies core mathematical theorems according to the set existence axioms needed to prove them.

Often the theorem is equivalent to the axiom. Hence the name “reverse mathematics.”

A large number of theorems fall into a small number of equivalence classes.

The equivalence classes correspond to benchmarks in Gödel’s hierarchy of consistency strength.

See my book [6] and my paper [7].

Some benchmarks in the Gödel hierarchy:

strong:

$\vdots$

supercompact cardinal

$\vdots$

measurable cardinal

$\vdots$

ZFC (Zermelo/Fraenkel set theory)

ZC (Zermelo set theory)

simple type theory

medium:

Z_2 (second-order arithmetic)

$\vdots$

Π^1_2-CA_0 (Π^1_2 comprehension)

Π^1_1-CA_0 (Π^1_1 comprehension)

ATR_0 (arithmetical transfinite recursion)

ACA_0 (arithmetical comprehension)

weak:

WKL_0 (weak König's lemma)

RCA_0 (recursive comprehension)

PRA (primitive recursive arithmetic)

EFA (elementary function arithmetic)

bounded arithmetic

$\vdots$

Toward objectivity in mathematics, I see two insights drawn from reverse mathematics.

1. The bulk of core mathematical theorems fall at the lowest levels.

   This suggests that higher set theory may be largely irrelevant.

2. The lowest levels are conservative over PRA (primitive recursive arithmetic).

   This leads to strong partial realizations of Hilbert's program. A large portion of core mathematics, sufficient for applications, is validated in a finitistically provable way. See my paper [5].

   This may open a path to objectivity in mathematics.

6. Wider cultural significance?

Historically, are trends in philosophy of mathematics parallel to trends in world culture?

Plato and Aristotle.

The Renaissance.

The Enlightenment.

The 19th century.

Early 20th century: intuitionism in philosophy of mathematics; subjectivism and collectivism in the wider culture.

Late 20th century: set-theoretic realism in philosophy of mathematics; religious fundamentalism in the wider culture.

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