Forcing With Trees and Conservation Results for WKL_0

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Two important subsystems of 2nd order arithmetic: RCA_0 and WKL_0

$\mathsf{RCA}_0 =$ basic axioms $+$
Δ^0_1 comprehension $+\, \Sigma^0_1$ induction

RCA_0 is conservative over PRA for Π^0_2 sentences.

The minimum ω-model of RCA_0 is the recursive sets.

$\mathsf{WKL}_0 = \mathsf{RCA}_0 +$ Weak König's Lemma:
every infinite subtree of the full binary tree has an infinite path.

WKL_0 is conservative over RCA_0 for Π^1_1 sentences.

The "hard core" of ω-models of WKL_0 is the recursive sets.

Foundational significance of these and related results:

1. Hilbert's program of finitistic reductionism.

Many mathematical theorems are finitistically reducible, because provable in WKL_0.

2. reverse mathematics.

RCA_0 and WKL_0 are two of the basic systems.

See my book.

Book on Reverse Mathematics:

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Perspectives in Mathematical Logic
Springer-Verlag, 1999
XIV + 445 pages

Web: `www.math.psu.edu/simpson/sosoa/`

Order: 1-800-SPRINGER

List price: $60

Discount: 30 percent for ASL members, mention promotion code S206

The FOM list:

FOM is an automated e-mail list for discussing foundations of mathematics. There are currently more than 400 subscribers. There have been more than 3400 postings.

FOM was created September 1997 by H. Friedman and S. Simpson.

FOM is maintained and moderated by S. Simpson.

FOM postings and information are available on the web at

`www.math.psu.edu/simpson/fom/`

The hierarchy of consistency strengths:

strong	supercompact cardinal ⋮ measurable cardinal ⋮ ZFC (ZF set theory with choice) Zermelo set theory
medium	Z_2 (2nd order arithmetic) ⋮ Π^1_2 comprehension Π^1_1 comprehension ATR_0 (arith. transfinite recursion) ACA_0 (arithmetical comprehension)
weak	WKL_0 (weak König's lemma) RCA_0 (recursive comprehension) PRA (primitive recursive arithmetic) EFA (elementary arithmetic) bounded arithmetic ⋮

“tree” = infinite subtree of the full binary tree.

“path” = infinite path.

Weak König’s Lemma says:
every “tree” has a “path”.

Let M be a countable model of RCA_0.

$\mathcal{T}_M = \{T \in M : M \models T \text{ is a tree}\}$

Force with $\mathcal{T}_M$.

Key lemma:
If G is a generic path, then $M(G)$ satisfies Σ^0_1 induction.

This leads to several conservation results.

Let M be a countable model of RCA_0.

Theorem 1 (Harrington, 1977).
There exists $M' \supseteq_\omega M$ (same 1st order part) such that M' is a model of WKL_0.

Proof: Adjoin generic paths. Σ^0_1 induction is preserved.

Theorem 2 (Tanaka, 1995).
M is the hard core of all such M'.

A consequence of Theorem 1 is:
WKL_0 is conservative over RCA_0 for Π^1_1 sentences, i.e., sentences of the form

$$\forall X\ \theta(X)$$

where $\theta(X)$ is arithmetical.

After proving Theorem 2, Tanaka conjectured:
WKL_0 is conservative over RCA_0 for sentences of the form

$$\forall X\ \exists \text{ unique } Y\ \theta(X,Y)$$

where $\theta(X,Y)$ is arithmetical.

Attempted proof of Tanaka's conjecture: (Tanaka, Yamazaki, Fernandes; 1999)

Find $M_1, M_2, M_3 \supseteq_\omega M$ such that $M_3 \supseteq M_1 \cup M_2$ and $M_1 \cap M_2 = M$ and $M_1, M_2, M_3 \models \mathsf{WKL}_0$.

This idea cannot work!

Dramatis personae:

Sacks (Harvard/MIT) $\rightarrow$
Friedman, Simpson, Harrington

Simpson (Penn State) $\rightarrow$ Ferreira (Lisbon)
$\rightarrow$ Fernandes

Harrington (Berkeley) $\rightarrow$ Tanaka (Tohoku)
$\rightarrow$ Yamazaki

Theorem (Simpson, 1999).
Let M satisfy Σ^0_1 induction and not Σ^0_2 induction. Then there is a pair of M-recursively enumerable, M-recursively inseparable sets such that, if X and Y are separating sets and (M, X, Y) satisfies Σ^0_1 induction, then the symmetric difference of X and Y is M-finite.

Proof: Formalize in RCA_0 the following result of Kučera 1986:

There is a pair of disjoint, recursively inseparable, r.e. sets B_1, B_2 such that if Z is the symmetric difference of any two separating sets, then either Z is finite or $Z \geq_T 0'$.

Simplified proof of Kučera's result:
Let A be an r.e. set such that, for all n,

$$\{n\}(n)\downarrow \quad \text{if and only if} \quad \{n\}_{a_n}(n)\downarrow$$

where $a_n =$ the nth element of $\overline{A}$. Then Friedberg-split $A = B_1 \cup B_2$.

Consequently, WKL_0 is *not* conservative over RCA_0 for sentences of the form

$$\exists \text{ unique-}* \; X \; \psi(X)$$

where $\psi(X)$ is Π^0_1.

Unique-$*$ means unique up to finite symmetric difference.

This refutes another conjecture of Tanaka.

Nevertheless, Tanaka's main conjecture is true!

Theorem (Simpson, 1999).
WKL_0 is conservative over RCA_0 for sentences of the form $\forall X \; \exists$ unique $Y \; \theta(X,Y)$ with $\theta(X,Y)$ arithmetical.

For θ Σ^0_3 this was first proved by Antonio Marques Fernandes.

Proof. Say that $T \in \mathcal{T}_M$ is *universal* if for all $T' \in \mathcal{T}_M$ there exists an M-recursive functional

$$\Phi : \text{paths in } T \ \rightarrow \ \text{paths in } T'.$$

Show that all universal trees force the same Σ^1_1 sentences. (Compare homogeneous forcing in set theory.) This gives the theorem.

Technical refinement: Say $T \in \mathcal{T}_M$ is *homogeneous* if for all $T', T'' \subseteq T$ in $\mathcal{T}_M$ there is an M-recursive homeomorphism

$$\Phi : \text{paths in } T' \ \leftrightarrow \ \text{paths in } T''.$$

This implies that T', T'' force the same sentences. Homogeneous universal trees exist, by Pour-El/Kripke. From this we get a stronger form of Tanaka's conjecture.