Weak Degrees of Π^0_1 Subsets of 2^ω

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June 16, 2005

Note: This is an abstract of an invited one-hour talk, to be given in a Recursion Theory workshop, part of the program *Computational Prospects of Infinity*, June 20 to August 15, 2005, at the Institute for Mathematical Sciences, National University of Singapore.

Abstract

Let $P, Q \subseteq 2^\omega$ be viewed as *mass problems*, i.e., "decision problems with more than one solution." We say that the mass problem P is *weakly reducible to* the mass problem Q if, for every solution Y of Q, there exists a solution X of P such that X is Turing reducible to Y. A *weak degree* is an equivalence class of mass problems under mutual weak reducibility. (Weak degrees are also known as *Muchnik degrees*.) Let $\mathcal{P}_w$ be the set of weak degrees of nonempty Π^0_1 subsets of 2^ω, partially ordered by weak reducibility. It is easy to see that $\mathcal{P}_w$ is a countable distributive lattice. The speaker and others have studied $\mathcal{P}_w$ in a series of publications beginning in 1999. Our principal findings are as follows.

1. There is a natural embedding of $\mathcal{R}_T$, the countable semilattice of recursively enumerable Turing degrees, into $\mathcal{P}_w$. This embedding is one-to-one and preserves the partial ordering $\leq$, the semilattice operation $\vee$, and the top and bottom elements $\mathbf{0}$ and $\mathbf{0}'$. We identify $\mathcal{R}_T$ with its image in $\mathcal{P}_w$ under this embedding.

2. Like the semilattice $\mathcal{R}_T$, the lattice $\mathcal{P}_w$ is structurally rich. In particular, any countable distributive lattice is lattice embeddable in any nontrivial initial segment of $\mathcal{P}_w$. Also, the $\mathcal{P}_w$ analog of the Sacks Splitting Theorem holds. These structural results are proved by means of priority arguments. The $\mathcal{P}_w$ analog of the Sacks Density Theorem remains as an open problem.

3. Unlike $\mathcal{R}_T$, the lattice $\mathcal{P}_w$ contains a large number of specific, natural degrees other than the top and bottom elements. These specific, natural degrees in $\mathcal{P}_w$ are related to foundationally interesting topics such as reverse mathematics, algorithmic randomness, subrecursive hierarchies from Gentzen-style proof theory, and computational complexity.

4. All of these specific, natural degrees in $\mathcal{P}_w$ are incomparable with all of the recursively enumerable Turing degrees in $\mathcal{R}_T$. The only exceptions are $\mathbf{0}$ and $\mathbf{0}'$, the top and bottom elements of $\mathcal{R}_T$, which are the same as the top and bottom elements of $\mathcal{P}_w$.