Mass Problems and Degrees of Unsolvability

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Motivation:

Recall that $\mathcal{E}_T$ is the upper semilattice of recursively enumerable Turing degrees.

Two basic, classical, unresolved issues concerning $\mathcal{E}_T$ are:

Issue 1: To find a specific, natural, r.e. Turing degree $\mathbf{a} \in \mathcal{E}_T$ which is $> \mathbf{0}$ and $< \mathbf{0}'$.

Issue 2: To find a "smallness property" of an infinite co-r.e. set $A \subseteq \omega$ which insures that $\deg_T(A) = \mathbf{a} \in \mathcal{E}_T$ is $> \mathbf{0}$ and $< \mathbf{0}'$.

These unresolved issues go back to Post's 1944 paper, *Recursively enumerable sets of positive integers and their decision problems*.

Mass Problems to the Rescue!

We address Issues 1 and 2 by passing from *decision problems* to *mass problems*.

Outline of this talk:

We embed $\mathcal{E}_T$ into a slightly larger structure, $\mathcal{P}_w$, which is much better behaved. In the $\mathcal{P}_w$ context, we obtain satisfactory, positive answers to Issues 1 and 2.

What is this wonderful structure $\mathcal{P}_w$?

Briefly, $\mathcal{P}_w$ is the lattice of weak degrees of mass problems associated with nonempty Π^0_1 subsets of 2^ω.

In order to explain $\mathcal{P}_w$, we must first explain:

- mass problems,
- weak degrees, and
- nonempty Π^0_1 subsets of 2^ω.

Mass problems (informal discussion):

A "decision problem" is the problem of deciding whether a given $n \in \omega$ belongs to a fixed set $A \subseteq \omega$ or not. To compare decision problems, we use Turing reducibility. $A \leq_T B$ means that A can be computed using an oracle for B.

A "mass problem" is a problem with a not necessarily unique solution. (By contrast, a "decision problem" has only one solution.)

The "mass problem" associated with a set $P \subseteq \omega^\omega$ is the "problem" of computing an element of P.

The "solutions" of P are the elements of P.

One mass problem is said to be "reducible" to another if, given any solution of the second problem, we can use it as an oracle to compute a solution of the first problem.

Rigorous definition:

Let P and Q be subsets of ω^ω.

We view P and Q as mass problems.

We say that P is *weakly reducible* to Q if

$$(\forall Y \in Q)\ (\exists X \in P)\ (X \leq_T Y)\ .$$

This is abbreviated $P \leq_w Q$.

Summary:

$P \leq_w Q$ means that, given any solution of Q, we can use it as an oracle to compute a solution of P.

Digression: weak vs. strong reducibility

Let P and Q be subsets of ω^ω.

1. P is *weakly reducible* to Q, $P \leq_w Q$, if for all $Y \in Q$ there exists e such that $\{e\}^Y \in P$.

2. P is *strongly reducible* to Q, $P \leq_s Q$, if there exists e such that $\{e\}^Y \in P$ for all $Y \in Q$.

Strong reducibility is a uniform variant of weak reducibility. By a result of Nerode, there is an analogy:

$$\frac{\text{weak reducibility}}{\text{Turing reducibility}} = \frac{\text{strong reducibility}}{\text{truth table reducibility}}.$$

In this talk we deal only with weak reducibility.

Historical note:

Weak reducibility is due to Muchnik 1963.

Strong reducibility is due to Medvedev 1955.

The lattice $\mathcal{P}_w$:

We focus on Π^0_1 subsets of 2^ω, i.e., $P = \{\text{paths through } T\}$ where T is a recursive subtree of $2^{<\omega}$, the full binary tree of finite sequences of 0's and 1's.

We define $\mathcal{P}_w$ to be the set of weak degrees of nonempty Π^0_1 subsets of 2^ω, ordered by weak reducibility.

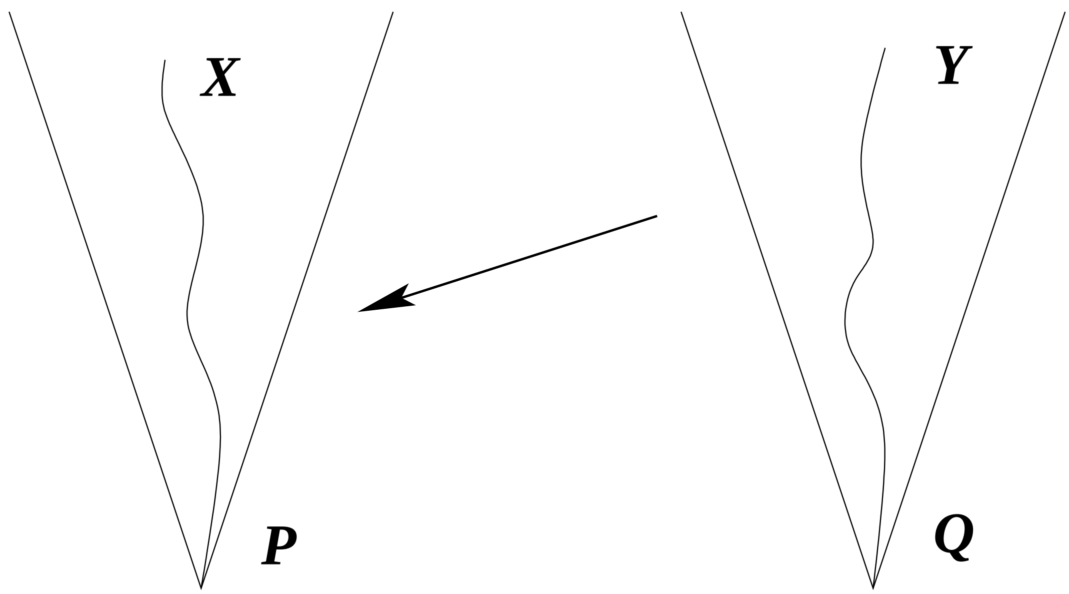
Basic facts about $\mathcal{P}_w$:

1. $\mathcal{P}_w$ is a distributive lattice, with l.u.b. given by $P \times Q = \{X \oplus Y \mid X \in P, Y \in Q\}$, and g.l.b. given by $P \cup Q$.

2. The bottom element of $\mathcal{P}_w$ is the weak degree of 2^ω.

3. The top element of $\mathcal{P}_w$ is the weak degree of PA $= \{\text{completions of Peano Arithmetic}\}$. (Scott/Tennenbaum).

Weak reducibility of Π^0_1 subsets of 2^ω:



$P \leq_w Q$ means:
$(\forall Y \in Q)\ (\exists X \in P)\ (X \leq_T Y)$.

P, Q are given by infinite recursive subtrees of the full binary tree of finite sequences of 0's and 1's.

X, Y are infinite (nonrecursive) paths through P, Q respectively.

The lattice $\mathcal{P}_w$ (review):

A *weak degree* is an equivalence class of subsets of ω^ω under the equivalence relation $P \leq_w Q$ and $Q \leq_w P$. The weak degrees have a partial ordering induced by $\leq_w$.

We define $\mathcal{P}_w$ to be the set of weak degrees of nonempty Π^0_1 subsets of 2^ω, partially ordered by weak reducibility.

$\mathcal{P}_w$ is a countable distributive lattice.

The bottom element of $\mathcal{P}_w$ is the weak degree of 2^ω.

The top element of $\mathcal{P}_w$ is the weak degree of

$$\text{PA} = \{\text{completions of Peano Arithmetic}\}.$$

Embedding $\mathcal{E}_T$ into $\mathcal{P}_w$:

Theorem (Simpson 2002)**:**
There is a natural embedding $\phi : \mathcal{E}_T \to \mathcal{P}_w$.

($\mathcal{E}_T =$ the semilattice of Turing degrees of r.e. subsets of ω. $\mathcal{P}_w =$ the lattice of weak degrees of nonempty Π^0_1 subsets of 2^ω.)

The embedding ϕ is given by

$$\phi : \text{deg}_T(A) \mapsto \text{deg}_w(\text{PA} \cup \{A\}).$$

Note that $\text{PA} \cup \{A\}$ is not a Π^0_1 set. However, it is of the same weak degree as a Π^0_1 set. This is already a nontrivial result.

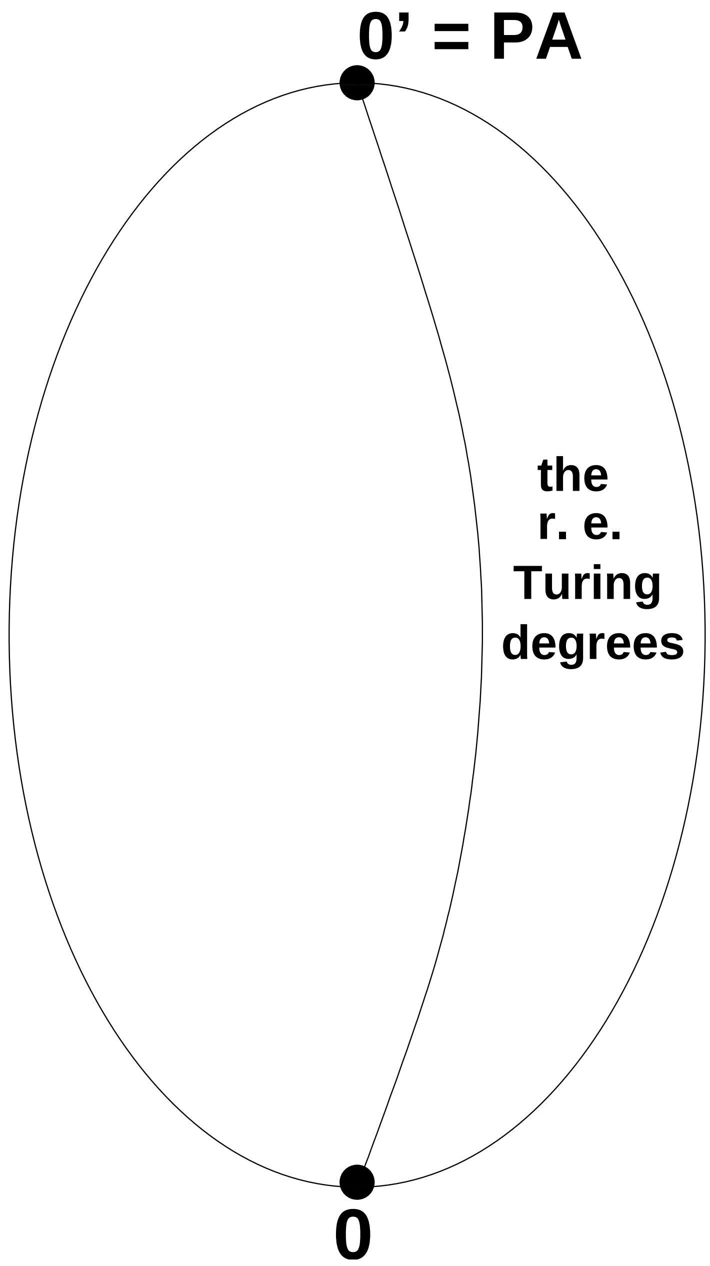
The embedding ϕ is one-to-one and preserves $\leq$, l.u.b., and the top and bottom elements.

Convention:

We identify $\mathcal{E}_T$ with its image in $\mathcal{P}_w$ under ϕ.

In particular, we identify $\mathbf{0}', \mathbf{0} \in \mathcal{E}_T$ with the top and bottom elements of $\mathcal{P}_w$.

A picture of the lattice $\mathcal{P}_w$:



$\mathcal{E}_T$ is embedded in $\mathcal{P}_w$. $\mathbf{0}'$ and $\mathbf{0}$ are the top and bottom elements of both $\mathcal{E}_T$ and $\mathcal{P}_w$.

Structural properties of $\mathcal{P}_w$:

1. $\mathcal{P}_w$ is a countable distributive lattice. Every countable distributive lattice is lattice embeddable in every initial segment of $\mathcal{P}_w$. (Binns/Simpson 2001)

2. The $\mathcal{P}_w$ analog of the Sacks Splittting Theorem holds. (Stephen Binns, 2002)

3. We conjecture that the $\mathcal{P}_w$ analog of the Sacks Density Theorem holds.

These structural results for $\mathcal{P}_w$ are proved by means of priority arguments, just as for $\mathcal{E}_T$.

4. Within $\mathcal{P}_w$ the degrees $\mathbf{r}_1$ and $\inf(\mathbf{r}_2, \mathbf{0}')$ are meet irreducible and do not join to $\mathbf{0}'$. (Simpson 2002, 2004)

5. $\mathbf{0}$ is meet irreducible. (This is trivial.)

Response to Issue 1:

Issue 1 was:

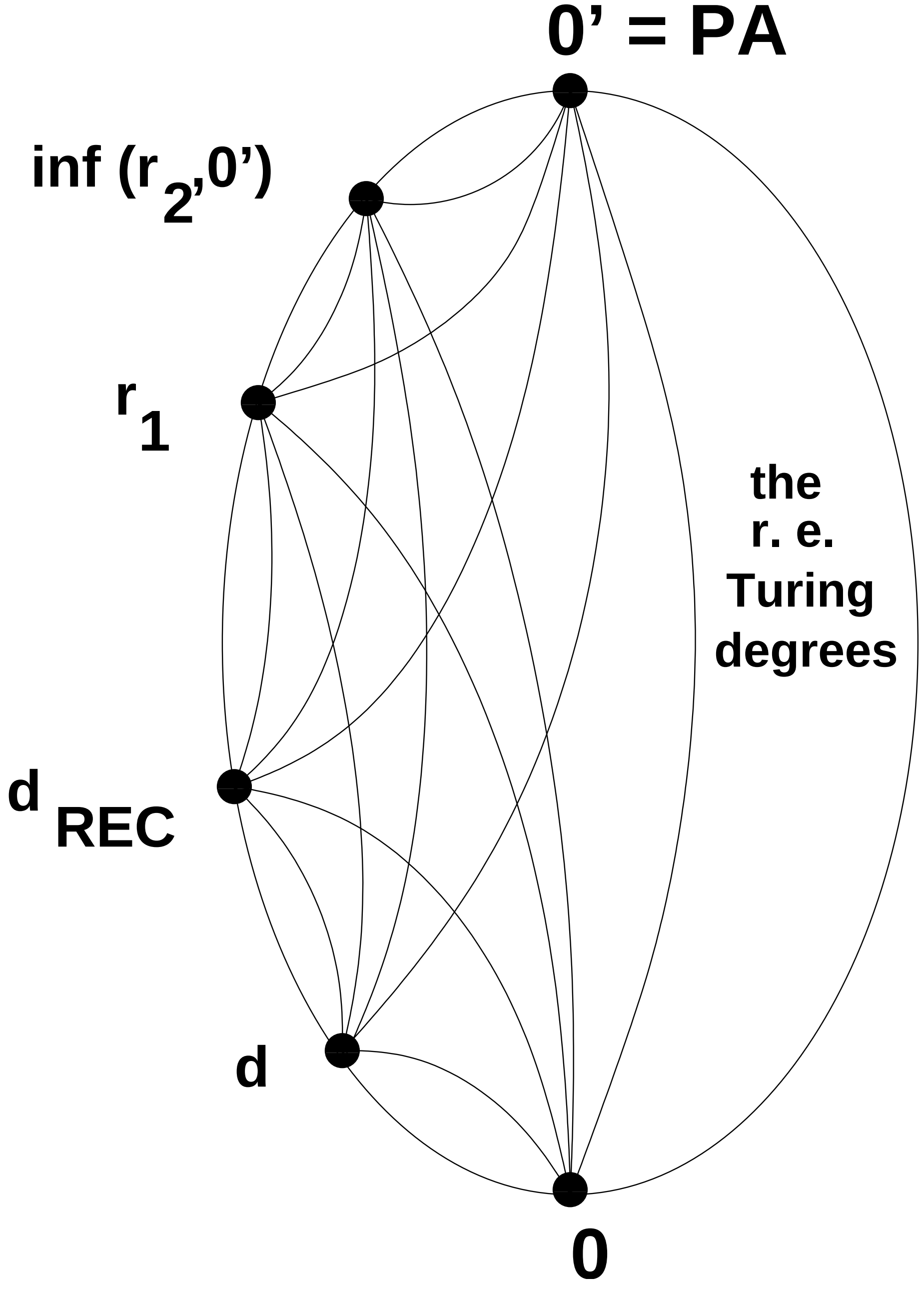
To find a specific, natural example of a recursively enumerable Turing degree which is $> \mathbf{0}$ and $< \mathbf{0}'$.

We do not know how to do this.

However, in the $\mathcal{P}_w$ context, we have discovered many specific, natural degrees which are $> \mathbf{0}$ and $< \mathbf{0}'$.

The specific, natural degrees in $\mathcal{P}_w$ which we have discovered are related to foundationally interesting topics:

- algorithmic randomness,
- diagonal nonrecursiveness,
- reverse mathematics,
- subrecursive hierarchies,
- computational complexity.



Note: Except for $\mathbf{0}'$ and $\mathbf{0}$, the r.e. Turing degrees are incomparable with these specific, natural degrees in $\mathcal{P}_w$.

Some specific, natural degrees in $\mathcal{P}_w$:

$\mathbf{r}_n =$ the weak degree of the set of n-random reals.

$\mathbf{d} =$ the weak degree of the set of diagonally nonrecursive functions.

$\mathbf{d}_{\mathsf{REC}} =$ the weak degree of the set of diagonally nonrecursive functions which are recursively bounded.

Theorem (Simpson 2002, Ambos $\cdots$ 2004)**:**
In $\mathcal{P}_w$ we have

$$\mathbf{0} < \mathbf{d} < \mathbf{d}_{\mathsf{REC}} < \mathbf{r}_1 < \inf(\mathbf{r}_2, \mathbf{0}') < \mathbf{0}'.$$

Theorem (Simpson 2004)**:**

1. $\mathbf{r}_1$ is the maximum weak degree of a Π^0_1 subset of 2^ω which is of positive measure.
2. $\inf(\mathbf{r}_2, \mathbf{0}')$ is the maximum weak degree of a Π^0_1 subset of 2^ω whose Turing upward closure is of positive measure.

Structural properties of $\mathcal{P}_w$:

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2. The $\mathcal{P}_w$ analog of the Sacks Splitting Theorem holds. (Stephen Binns, 2002)

3. We conjecture that the $\mathcal{P}_w$ analog of the Sacks Density Theorem holds.

These structural results for $\mathcal{P}_w$ are proved by means of priority arguments, just as for $\mathcal{E}_T$.

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Another source of specific degrees in $\mathcal{P}_w$: almost everywhere domination.

Definition (Dobrinen/Simpson 2004)**:**

B is *almost everywhere dominating* if, for almost all $X \in 2^\omega$, each function $\leq_T X$ is dominated by some function $\leq_T B$.

Here "almost all" refers to the fair coin measure on 2^ω.

Randomness and a. e. domination are closely related to the reverse mathematics of measure theory.

Some additional, natural degrees in $\mathcal{P}_w$:

Let $\mathbf{b}_1 = \deg_w(\mathsf{AED})$ where

$$\mathsf{AED} = \{B \mid B \text{ is a. e. dominating}\}.$$

Let $\mathbf{b}_2 = \deg_w(\mathsf{AED} \times \mathsf{R}_1)$ where

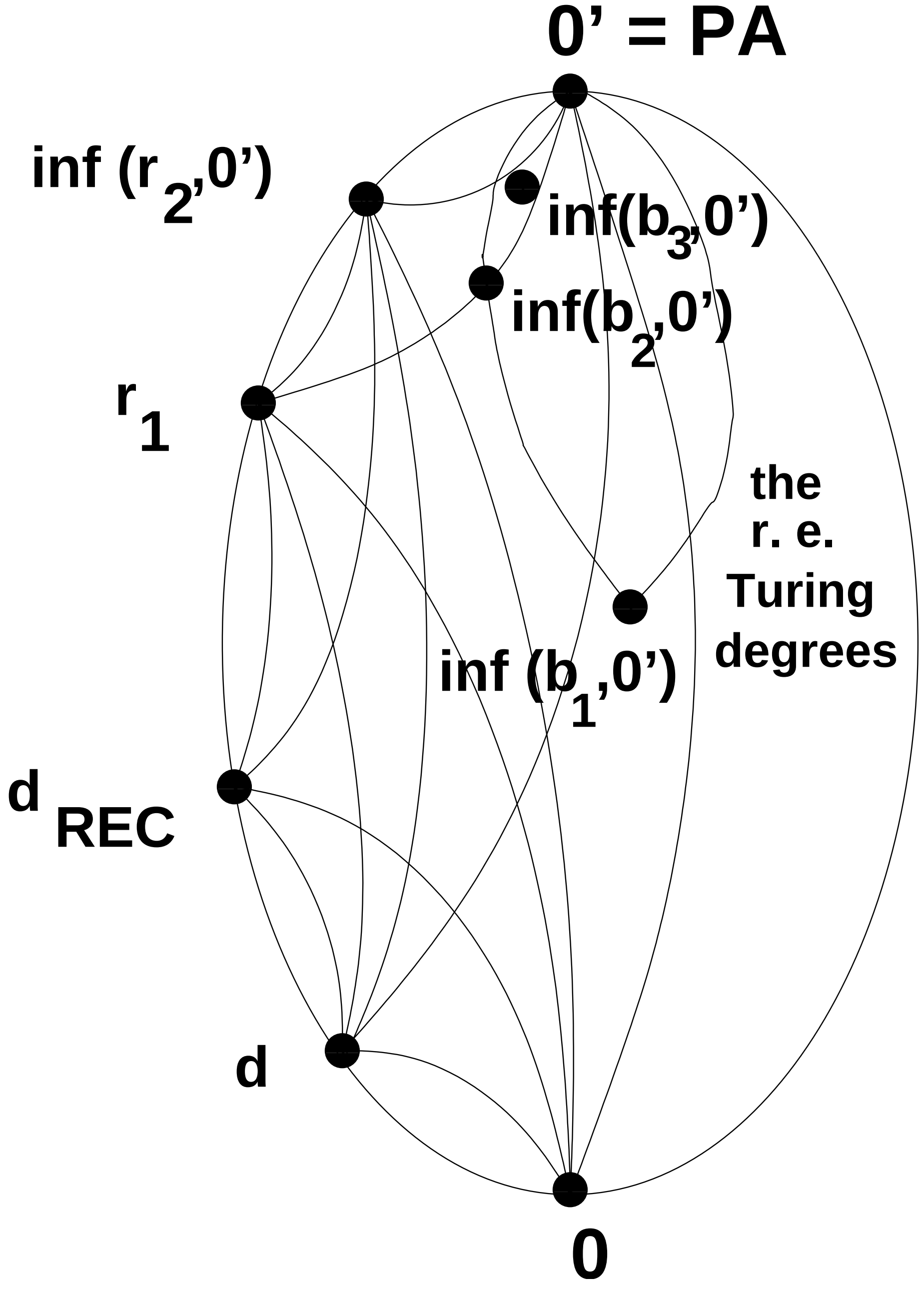
$$\mathsf{R}_1 = \{A \mid A \text{ is 1-random}\}.$$

Let $\mathbf{b}_3 = \deg_w(\mathsf{AED} \cap \mathsf{R}_1)$.

Theorem (2006)**:** In $\mathcal{P}_w$ we have:

- $\mathbf{0} < \inf(\mathbf{b}_1, \mathbf{0}') < \inf(\mathbf{b}_2, \mathbf{0}') < \inf(\mathbf{b}_3, \mathbf{0}') < \mathbf{0}'$.
- $\inf(\mathbf{b}_1, \mathbf{0}') <$ some r.e. degrees $< \mathbf{0}'$.
- $\inf(\mathbf{b}_2, \mathbf{0}') \mid$ all r.e. degrees except $\mathbf{0}, \mathbf{0}'$.
- $\inf(\mathbf{b}_3, \mathbf{0}') >$ some r.e. degrees $> \mathbf{0}$.

The proof uses virtually everything that is known about randomness and almost everywhere domination (Cholak, Greenberg, Hirschfeldt, Kjos-Hanssen, Miller, Nies, . . .).



Note that inf$(\mathbf{b}_1, \mathbf{0}')$ and inf$(\mathbf{b}_3, \mathbf{0}')$, unlike inf$(\mathbf{b}_2, \mathbf{0}')$, are comparable with some r.e. Turing degrees other than $\mathbf{0}'$ and $\mathbf{0}$.

Some additional, specific degrees in $\mathcal{P}_w$:

$\mathbf{d}_\alpha =$ the weak degree of the set of diagonally nonrecursive functions which are bounded by a recursive function at level α of the Wainer hierarchy. Here α is any ordinal $\leq \varepsilon_0$.

$\mathbf{d}^2 =$ the weak degree of the set of $f \oplus g$ such that f is diagonally nonrecursive, and g is diagonally nonrecursive relative to f. More generally, define $\mathbf{d}^n$ for all $n \geq 1$.

Theorem (Simpson 2004, Ambos $\cdots$ 2004)**:**
In $\mathcal{P}_w$ we have

$$\mathbf{r}_1 > \mathbf{d}_0 > \mathbf{d}_1 > \cdots > \mathbf{d}_\alpha > \cdots > \mathbf{d}_{\mathsf{REC}}$$

and

$$\mathbf{d} = \mathbf{d}^1 < \mathbf{d}^2 < \cdots < \mathbf{d}^n < \cdots < \mathbf{r}_1 \,.$$

We conjecture that $\mathbf{d}^n$ is incomparable with $\mathbf{d}_\alpha$ and with $\mathbf{d}_{\mathsf{REC}}$. This would be the first example of specific, natural degrees in $\mathcal{P}_w$ which are incomparable with each other.

The Embedding Lemma:

If $S \subseteq \omega^\omega$ *is* Σ^0_3 *and if* $P \subseteq 2^\omega$ *is nonempty* Π^0_1*, then* $\deg_w(S \cup P) \in \mathcal{P}_w$.

It follows that, for many Σ^0_3 sets $S \subseteq \omega^\omega$, $\deg_w(S) \in \mathcal{P}_w$.

Examples:

1. $\mathsf{R}_1 = \{X \in 2^\omega \mid X$ is 1-random$\}$.

Since R_1 is Σ^0_2, it follows by the Embedding Lemma that $\mathbf{r}_1 = \deg_w(R_1) \in \mathcal{P}_w$.

2. $\mathsf{R}_2 = \{X \in 2^\omega \mid X$ is 2-random$\}$.

Since R_2 is Σ^0_3, it follows by the Embedding Lemma that $\inf(\mathbf{r}_2, \mathbf{0}') = \deg_w(R_2 \cup \mathsf{PA}) \in \mathcal{P}_w$.

3. $\mathsf{D} = \{f \in \omega^\omega \mid f$ is diagonally nonrecursive$\}$.

Since D is Π^0_1, $\mathbf{d} = \deg_w(D) \in \mathcal{P}_w$.

4. $\mathsf{D}_{\mathsf{REC}} = \{f \in D \mid f$ is recursively bounded$\}$.

Since $\mathsf{D}_{\mathsf{REC}}$ is Σ^0_3, $\mathbf{d}_{\mathsf{REC}} = \deg_w(\mathsf{D}_{\mathsf{REC}}) \in \mathcal{P}_w$.

5. Let $A \subseteq \omega$ be r.e. Since $\{A\}$ is Π^0_2, $\deg_w(\{A\} \cup \mathsf{PA}) \in \mathcal{P}_w$. This gives our embedding of $\mathcal{E}_T$ into $\mathcal{P}_w$.

The Embedding Lemma (restated):

Let $S \subseteq \omega^\omega$ be Σ^0_3. Let $P \subseteq 2^\omega$ be nonempty Π^0_1. Then $\exists$ nonempty Π^0_1 $Q \subseteq 2^\omega$ such that $Q \equiv_w S \cup P$.

Proof (sketch). **Step 1.** By Skolem functions, we may assume that $S \subseteq \omega^\omega$ is Π^0_1.

Step 2. We have $S = \{\text{paths through } T_S\}$, $P = \{\text{paths through } T_P\}$, where T_S, T_P are recursive subtrees of $\omega^{<\omega}$, $2^{<\omega}$ respectively. May assume $\tau(n) \geq 2$ for all $n < |\tau|$, $\tau \in T_S$.

Define $Q = \{\text{paths through } T_Q\}$, where T_Q is the set of all $\rho \in \omega^{<\omega}$ of the form

$$\rho = \sigma_0{}^\frown\langle m_0\rangle^\frown\sigma_1{}^\frown\langle m_1\rangle^\frown\cdots{}^\frown\langle m_{k-1}\rangle^\frown\sigma_k$$

where

- $\sigma_0, \sigma_1, \ldots, \sigma_k \in T_P$,
- $\langle m_0, m_1, \ldots, m_{k-1}\rangle \in T_S$,
- $\rho(n) \leq \max(n, 2)$ for all $n < |\rho|$.

One can show that $Q \equiv_w S \cup P$.

Step 3. Q is Π^0_1 and recursively bounded. Hence, we can find Π^0_1 $Q^* \subseteq 2^\omega$ such that Q^* is recursively homeomorphic to Q. Done.

Response to Issue 2:

Issue 2 was:

To find a "smallness property" of an infinite Π^0_1 (i.e., co-r.e.) set $A \subseteq \omega$ which insures that the Turing degree of A is $> \mathbf{0}$ and $< \mathbf{0}'$.

We do not know how to do this.

However, in the $\mathcal{P}_w$ context, we have identified several "smallness properties" of a Π^0_1 set $P \subseteq 2^\omega$ which insure that the weak degree of P is $> \mathbf{0}$ and $< \mathbf{0}'$.

One result of this type:

Theorem (Simpson 2002)**:**

Let $\mathbf{p}$ be the weak degree of a Π^0_1 set $P \subseteq 2^\omega$ which is <u>thin</u> and <u>perfect</u>. Then $\mathbf{p}$ is incomparable with $\mathbf{r}_1$. Hence $\mathbf{0} < \mathbf{p} < \mathbf{0}'$.

Background on thin Π^0_1 sets:

Definition:

A Π^0_1 set $P \subseteq 2^\omega$ is said to be *thin* if, for all Π^0_1 sets $Q \subseteq P$, $P \setminus Q$ is Π^0_1.

Thin perfect Π^0_1 subsets of 2^ω have been constructed by means of priority arguments. Much is known about them. For example, any two such sets are automorphic in the lattice of Π^0_1 subsets of 2^ω under inclusion.

(Martin/Pour-El 1970,
Downey/Jockusch/Stob 1990, 1996,
Cholak et al 2001)

Some additional "smallness properties":

Let P be a nonempty Π^0_1 subset of 2^ω.

1. P is *small* if there is no recursive function f such that for all n there exist n members of P which differ at level $f(n)$ in the binary tree. (Binns 2003)

Example: Let $A \subseteq \omega$ be hypersimple, and let $A = B_1 \cup B_2$ where B_1, B_2 are r.e. Then $P = \{X \in 2^\omega \mid X \text{ separates } B_1, B_2\}$ is small.

Theorem (Binns)**:**

If P is small, the weak degree of P is $< \mathbf{0}'$.

2. P is *h-small* if there is no recursive, canonically indexed sequence of pairwise disjoint clopen sets D_n, $n \in \omega$, such that $P \cap D_n \neq \emptyset$ for all n. (Simpson 2003)

Theorem (Simpson)**:**

If P is h-small, the weak degree of P is $< \mathbf{0}'$.

Summary of this talk:

There are basic, unresolved issues concerning $\mathcal{E}_T$, the semilattice of recursively enumerable Turing degrees. One of the issues is the lack of specific, natural, r.e. degrees.

We embed $\mathcal{E}_T$ into $\mathcal{P}_w$, the lattice of weak degrees of nonempty Π^0_1 subsets of 2^ω. We identify $\mathcal{E}_T$ with its image in $\mathcal{P}_w$.

In the $\mathcal{P}_w$ context, some of the unresolved issues can be satisfactorily addressed.

In particular, $\mathcal{P}_w$ contains many specific, natural degrees which are related to foundationally interesting topics:

- algorithmic randomness,
- reverse mathematics,
- computational complexity.

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Some of my papers are available at

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Transparencies for my talks are available at

`http://www.math.psu.edu/simpson/talks/`.

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