DEGREES OF UNSOLVABILITY AND SYMBOLIC DYNAMICS

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Degrees of unsolvability.

Let $\mathbb{N} = \{\text{the natural numbers}\} = \{0, 1, 2, \ldots\}$.
Let $\Omega = \{0,1\}^{\mathbb{N}} = \{x \mid x : \mathbb{N} \to \{0,1\}\}$
= the Cantor space.

Consider a Turing machine M with three infinite tapes: the input tape, the output tape, and the scratch tape. Assume that the input tape is read-only, and the output tape is write-once. We then use M to define a functional $\Phi_M :\subseteq \Omega \to \Omega$, as follows.

Given $x, y \in \Omega$ let $M(x)$ = the run of M starting with x on the input tape and blanks on the output and scratch tapes. We define $\Phi_M(x) = y$ if and only if $M(x)$ writes y on the output tape. Otherwise $\Phi_M(x)$ is undefined.

Here x is used as an "oracle" which helps us to compute y. We say that y is *computable relative to* x. This idea came from Turing.

Note that Φ_M is continuous on its domain.

For sets $P, Q \subseteq \Omega$ we define:

$P \geq_{\mathsf{s}} Q$, i.e., Q is *strongly reducible* to P, if and only if $\exists M \, \forall x \, (x \in P \Rightarrow \Phi_M(x) \in Q)$. In other words, $\Phi_M \upharpoonright P : P \to Q$.

$P \geq_{\mathsf{w}} Q$,i.e., Q is *weakly reducible* to P, if and only if $\forall x \, \exists M \, (x \in P \Rightarrow \Phi_M(x) \in Q)$.

Motivation: The sets $P, Q \subseteq \Omega$ are regarded as "problems." The "solutions" of P are just the elements of the set P. Such problems are known as *mass problems*. The problem P is said to be "solvable" if at least one of its solutions is computable. Otherwise P is said to be "unsolvable." The problem Q is said to be "reducible" to the problem P if each solution x of P can be used as an oracle to compute some solution y of Q.

The distinction between $\geq_{\mathsf{s}}$ and $\geq_{\mathsf{w}}$ lies in whether or not the Turing machine M which computes y relative to x is required to be independent of x.

History:

Kolmogorov 1932 developed his "calculus of problems" as a nonrigorous yet compelling explanation of Brouwer's intuitionism. Later Medvedev 1955 and Muchnik 1963 proposed strong and weak reducibility as rigorous explications of Kolmogorov's idea.

Some references:

Stephen G. Simpson, Mass problems and randomness, Bulletin of Symbolic Logic, 11, 2005, pages 1–27.

Stephen G. Simpson, Medvedev degrees of 2-dimensional subshifts of finite type, 8 pages, 1 May 2007; accepted 26 September 2007 for publication in Ergodic Theory and Dynamical Systems.

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Stephen G. Simpson, Mass problems and measure-theoretic regularity, Bulletin of Symbolic Logic, 15, 2009, pages 385–409.

More definitions:

$P \equiv_{\mathsf{s}} Q \Leftrightarrow (P \leq_{\mathsf{s}} Q \wedge Q \leq_{\mathsf{s}} P)$.

$P \equiv_{\mathsf{w}} Q \Leftrightarrow (P \leq_{\mathsf{w}} Q \wedge Q \leq_{\mathsf{w}} P)$.

$\mathrm{deg}_{\mathsf{s}}(P) = \{Q \mid P \equiv_{\mathsf{s}} Q\}$
= the *strong degree* of P.

$\mathrm{deg}_{\mathsf{w}}(P) = \{Q \mid P \equiv_{\mathsf{w}} Q\}$
= the *weak degree* of P.

$\mathcal{D}_{\mathsf{s}} = \{\mathrm{deg}_{\mathsf{s}}(P) \mid P \subseteq \Omega\}$.
$\mathcal{D}_{\mathsf{s}}$ has a partial ordering $\leq$ induced by $\leq_{\mathsf{s}}$.

$\mathcal{D}_{\mathsf{w}} = \{\mathrm{deg}_{\mathsf{w}}(P) \mid P \subseteq \Omega\}$.
$\mathcal{D}_{\mathsf{w}}$ has a partial ordering $\leq$ induced by $\leq_{\mathsf{w}}$.

Remark: Medvedev 1955 and Muchnik 1963 respectively noted that $\mathcal{D}_{\mathsf{s}}$ and $\mathcal{D}_{\mathsf{w}}$ are complete Brouwerian lattices. Aspects of these lattices have been studied by Dyment, Skvortsova, Sorbi, Terwijn, and others.

Note: The cardinality of $\mathcal{D}_{\mathsf{s}}$ and $\mathcal{D}_{\mathsf{w}}$ is $2^{2^{\aleph_0}}$.

Yet more definitions:

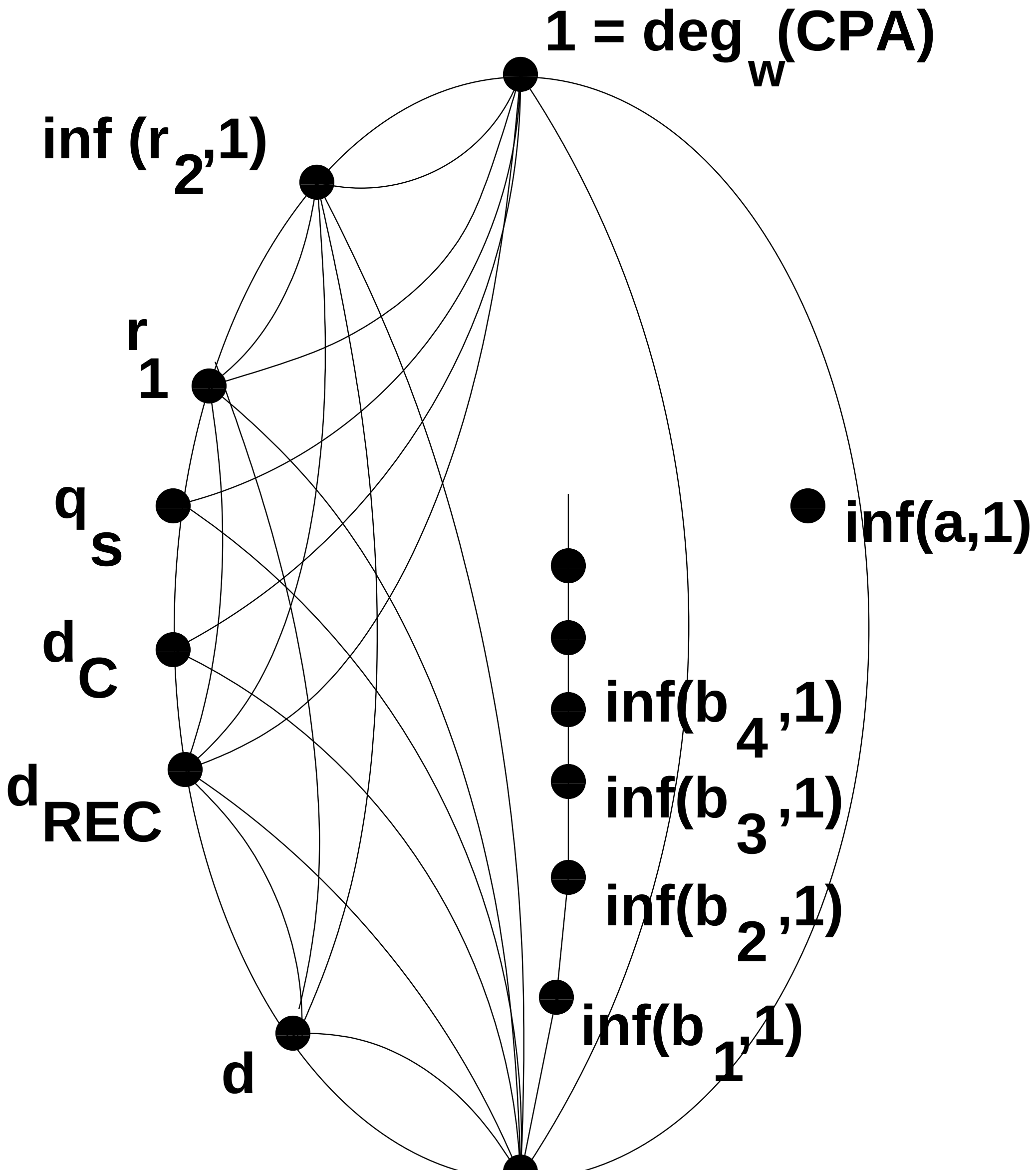
$U \subseteq \Omega$ is *effectively open* if there exists a comptable function $s : \mathbb{N} \to \{0,1\}^*$ such that $U = \bigcup_{n=0}^{\infty} \Omega_{s(n)}$.

$P \subseteq \Omega$ is *effectively closed* if $\Omega \setminus P$ is effectively open.

$\mathcal{E}_{\mathsf{s}} = \{\deg_{\mathsf{s}}(P) \mid \emptyset \neq P \subseteq \Omega,\ P \text{ eff. closed}\}$.

$\mathcal{E}_{\mathsf{w}} = \{\deg_{\mathsf{w}}(P) \mid \emptyset \neq P \subseteq \Omega,\ P \text{ eff. closed}\}$.

Remark: $\mathcal{E}_{\mathsf{s}}$ and $\mathcal{E}_{\mathsf{w}}$ are countable sublattices of $\mathcal{D}_{\mathsf{s}}$ and $\mathcal{D}_{\mathsf{w}}$ respectively. Much is known about them. For instance, both $\mathcal{E}_{\mathsf{s}}$ and $\mathcal{E}_{\mathsf{w}}$ contain a bottom degree, denoted $\mathbf{0}$, and a top degree, denoted $\mathbf{1}$. Obviously $\mathbf{0} = \deg_{\mathsf{s}}(\Omega) = \deg_{\mathsf{w}}(\Omega)$. However, the existence of $\mathbf{1}$ in $\mathcal{E}_{\mathsf{s}}$ and $\mathcal{E}_{\mathsf{w}}$ is not so obvious. A well-known characterization of $\mathbf{1}$ will be mentioned later.



A picture of $\mathcal{E}_{\mathrm{w}}$. Each black dot except $\inf(\mathbf{a},\mathbf{1})$ represents a specific, natural degree in $\mathcal{E}_{\mathrm{w}}$. As time permits we shall explain some of these degrees.

Symbolic dynamics.

Let A be a finite set of symbols. Let $\mathbb{Z} =$ the integers $= \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
We write $A^{\mathbb{Z}} = \{x \mid x : \mathbb{Z} \to A\}$.
This is the *full shift space* on A.
The *shift operator* $S : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is given by $S(x)(i) = x(i+1)$ for all $i \in \mathbb{Z}$.

A *subshift* is a set $X \subseteq A^{\mathbb{Z}}$ which is closed, nonempty, and *shift invariant*, i.e., $\forall x\, (x \in X \Leftrightarrow S(x) \in X)$.

If X and Y are subshifts, a *shift morphism* from X to Y is a continuous mapping $f : X \to Y$ such that $f(S(x)) = S(f(x))$ for all $x \in X$.

Given $E \subseteq A^* = \bigcup_{n=0}^{\infty} A^n$ let $X_E =$

$\{x \in A^{\mathbb{Z}} \mid (\forall i \in \mathbb{Z})\, \forall n\, \langle x(i+1), \ldots, x(i+n) \rangle \notin E\}$.

Thus E is a set of "excluded words."

Clearly X_E is a subshift, provided it is $\neq \emptyset$.

Moreover, all subshifts are of this form.

If E is finite, the subshift X_E
is said to be *of finite type*.

If E is computable, the subshift X_E
is said to be *of computable type*.

Some easy remarks:

1. If $f : X \to Y$ is a shift morphism, then $X \geq_{\mathsf{s}} Y$ and $X \geq_{\mathsf{w}} Y$.

In fact, f is a "block code."

2. If $f, f^{-1} : X \leftrightarrow Y$ are shift morphisms, then $X \equiv_{\mathsf{s}} Y$ and $X \equiv_{\mathsf{w}} Y$.

In other words, the strong and weak degrees of a subshift are "conjugacy invariants."

3. X is of computable type if and only if X is effectively closed.

4. If X is of computable type, then $\deg_{\mathsf{s}}(X) \in \mathcal{E}_{\mathsf{s}}$ and $\deg_{\mathsf{w}}(X) \in \mathcal{E}_{\mathsf{w}}$.

Theorem (Joseph Miller)**.** Conversely, each degree in $\mathcal{E}_{\mathsf{s}}$ or $\mathcal{E}_{\mathsf{w}}$ is, respectively, the strong or weak degree of a subshift of computable type.

The proof is ingenious but not difficult.

We now generalize to d-dimensional subshifts. For $d \geq 1$ we write $\mathbb{Z}^d = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{d}$.

As before, let A be a finite set of symbols. The *full d-dimensional shift space* over A is $A^{\mathbb{Z}^d} = \{x \mid x : \mathbb{Z}^d \to A\}$. The *shift operators* $S_k : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ for $k = 1, \ldots, d$ are given by $S_k(x)(i_1, \ldots, i_d) = x(i_1, \ldots, i_k + 1, \ldots, i_d)$.

A *d-dimensional subshift* is a set $X \subseteq A^{\mathbb{Z}^d}$ which is closed, nonempty, and *shift invariant*, i.e., $(\forall k)_{1 \leq k \leq d} \forall x \, (x \in X \Leftrightarrow S_k(x) \in X)$.

Each d-dimensional subshift is of the form
$X_E = \{x \in A^{\mathbb{Z}^d} \mid \forall n \, (\forall i_1, \ldots, i_d \in \mathbb{Z})$
$(\langle x(i_1 + j_1, \ldots, i_d + j_d) \rangle_{1 \leq j_1, \ldots, j_d \leq n} \notin E)\}$
where $E \subseteq \bigcup_{n=0}^{\infty} A^{\{1,\ldots,n\}^d}$. Thus E is a set of "excluded d-dimensional words."

If E is finite, we say that X_E is *of finite type*.

If E is computable, we say that X_E is *of computable type*.

All of our earlier remarks about the 1-dimensional case extend easily to the d-dimensional case.

Theorem (Simpson). Each degree in $\mathcal{E}_{\mathsf{s}}$ or $\mathcal{E}_{\mathsf{w}}$ is, respectively, the strong or weak degree of a 2-dimensional subshift of finite type.

The proof uses techniques going back to Berger 1965 and R. Robinson 1972. Another proof can be obtained by means of "self-replicating tile sets" (Durand/Romashchenko/Shen).

Remark. There are many specific, interesting degrees in $\mathcal{E}_{\mathsf{w}}$. By the above theorems, each such degree is realized by a 1-dimensional subshift of computable type (Miller) and by a 2-dimensional subshift of finite type (Simpson).

A possibly interesting research program:

Given a subshift X, explore the relationship between X's *dynamical properties* and X's *degree of unsolvability*, i.e., $\deg_{\mathsf{s}}(X)$ or $\deg_{\mathsf{w}}(X)$.

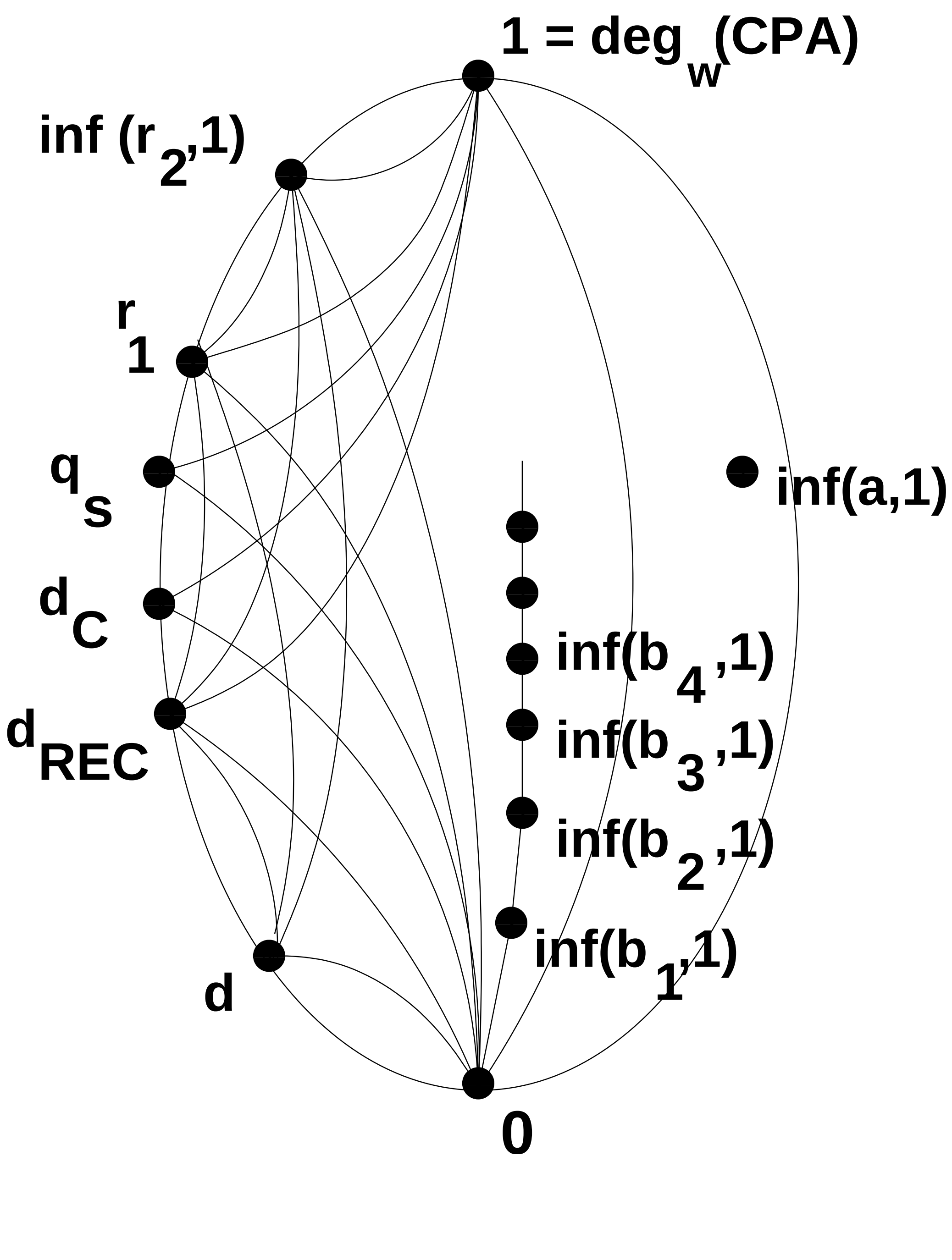
For example, the *entropy* of X is a well-known dynamical property which serves as an *upper bound* on the complexity of orbits. In particular $h(X) > 0$ implies $(\exists x \in X)\,(x$ is not computable$)$.

By contrast, the degree of unsolvability of X serves as a *lower bound* on the complexity of orbits. E.g., $\deg_{\mathsf{s}}(X) > \mathbf{0} \Leftrightarrow \deg_{\mathsf{w}}(X) > \mathbf{0} \Leftrightarrow$ $(\forall x \in X)\,(x$ is not computable$)$.

Theorem (Hochman)**.** If X is of computable type and *minimal* (i.e., every orbit is dense), then $\deg_{\mathsf{s}}(X) = \deg_{\mathsf{w}}(X) = \mathbf{0}$.

The proof is not difficult.

We finish by explaining some degrees in $\mathcal{E}_{\mathsf{w}}$.



A picture of $\mathcal{E}_{\mathrm{w}}$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{r} =$ randomness, $\mathbf{b} =$ LR-reducibility, $\mathbf{q} =$ dimension, $\mathbf{d} =$ diagonal nonrecursiveness.

We now explain some degrees in $\mathcal{E}_{\mathsf{w}}$.

The top degree in $\mathcal{E}_{\mathsf{w}}$ is $\mathbf{1} = \deg_{\mathsf{w}}(\mathsf{CPA})$ where CPA is the problem of finding a complete, consistent theory which includes first-order arithmetic.

We also have $\inf(\mathbf{a}, \mathbf{1}) \in \mathcal{E}_{\mathsf{w}}$ where $\mathbf{a}$ is any recursively enumerable Turing degree.

We have $\mathbf{d} \in \mathcal{E}_{\mathsf{w}}$ where $\mathbf{d} = \deg_{\mathsf{w}}(\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is *diagonally nonrecursive*$\})$, i.e., $\forall n\,(f(n) \neq \varphi_n(n))$.

Let REC $= \{g \in \mathbb{N}^{\mathbb{N}} \mid g$ is recursive$\}$. Let C be any “nice” subclass of REC. For instance $C =$ REC, or $C = \{g \in$ REC $\mid g$ is primitive recursive$\}$. We have $\mathbf{d}_C \in \mathcal{E}_{\mathsf{w}}$ where $\mathbf{d}_C = \deg_{\mathsf{w}}(\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is diagonally nonrecursive and C-*bounded*$\})$, i.e., $(\exists g \in C)\,\forall n\,(f(n) < g(n))$.

Also, $\mathbf{d}_C = \deg_{\mathsf{w}}(\{z \in \Omega \mid z$ is *C-complex*$\}$,
i.e., $(\exists g \in C)\, \forall n\, (\mathsf{K}(z \restriction \{1, \ldots, g(n)\}) \geq n)\})$.
Moreover, $\mathbf{d}_{C'} < \mathbf{d}_C$ whenever C' contains a function which dominates all functions in C.

For $z \in \Omega$ let $\dim(z) =$ the *effective Hausdorff dimension* of x, i.e.,

$$\dim(z) = \liminf_{n \to \infty} \frac{\mathsf{K}(z \restriction \{1, \ldots, n\})}{n}.$$

Given a right recursively enumerable real number s, we have $\mathbf{q}_s \in \mathcal{E}_{\mathsf{w}}$ where
$\mathbf{q}_s = \deg_{\mathsf{w}}(\{z \mid \dim(z) > s\})$.
Moreover, $s < t$ implies $\mathbf{q}_s < \mathbf{q}_t$ (Miller).

We have $\mathbf{r}_1 \in \mathcal{E}_{\mathsf{w}}$ where $\mathbf{r}_1 = \deg_{\mathsf{w}}(\{z \in \Omega \mid$ z is random in the sense of Martin-Löf$\})$.

We also have $\inf(\mathbf{r}_2, \mathbf{1}) \in \mathcal{E}_{\mathsf{w}}$ where
$\mathbf{r}_2 = \deg_{\mathsf{w}}(\{z \in \Omega \mid z$ is random relative to the halting problem$\})$.

Using x as an oracle, define

$\mathsf{R}^x = \{z \in \Omega \mid z \text{ is random relative to } x\}$

and $\mathsf{K}^x(n) =$ the prefix-free Kolmogorov complexity of n relative to x.

Define $x \leq_{\mathsf{LR}} y \Leftrightarrow \mathsf{R}^y \subseteq \mathsf{R}^x$

and $x \leq_{\mathsf{LK}} y \Leftrightarrow \exists c \forall n\, (\mathsf{K}^y(n) \leq \mathsf{K}^x(n) + c)$.

Theorem (Miller/Kjos-Hanssen/Solomon)**.** We have $x \leq_{\mathsf{LR}} y$ if and only if $x \leq_{\mathsf{LK}} y$.

For each recursive ordinal number α, let $0^{(\alpha)} =$ the αth iterated Turing jump of 0. Thus $0^{(1)} =$ the halting problem, and $0^{(\alpha+1)} =$ the halting problem relative to $0^{(\alpha)}$, etc. This is the hyperarithmetical hierarchy. We embed it naturally into $\mathcal{E}_{\mathsf{w}}$ as follows.

Theorem (Simpson)**.** $0^{(\alpha)} \leq_{\mathsf{LR}} y \Leftrightarrow$ every $\Sigma^0_{\alpha+2}$ set includes a $\Sigma^{0,y}_2$ set of the same measure. Moreover, letting $\mathbf{b}_\alpha = \deg_{\mathsf{w}}(\{y \mid 0^{(\alpha)} \leq_{\mathsf{LR}} y\})$ we have $\inf(\mathbf{b}_\alpha, \mathbf{1}) \in \mathcal{E}_{\mathsf{w}}$ and $\inf(\mathbf{b}_\alpha, \mathbf{1}) < \inf(\mathbf{b}_{\alpha+1}, \mathbf{1})$.

References:

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THE END.

THANK YOU!