Reverse Mathematics and Π^1_2 Comprehension

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Abstract

This is joint work with Carl Mummert. We initiate the reverse mathematics of general topology. We show that a certain metrization theorem is equivalent to Π^1_2 comprehension. An *MF space* is defined to be a topological space of the form $\mathrm{MF}(P)$ with topology generated by $\{N_p \mid p \in P\}$. Here P is a poset, $\mathrm{MF}(P)$ is the set of maximal filters on P, and $N_p = \{F \in \mathrm{MF}(P) \mid p \in F\}$. If the poset P is countable, the space $\mathrm{MF}(P)$ is said to be *countably based*. The class of countably based MF spaces can be defined and discussed within the subsystem ACA_0 of second-order arithmetic. One can prove within ACA_0 that every complete separable metric space is homeomorphic to a countably based MF space which is regular. We show that the converse statement, "every countably based MF space which is regular is homeomorphic to a complete separable metric space," is equivalent to Π^1_2-CA_0. The equivalence is proved in the weaker system Π^1_1-CA_0. This is the first example of a theorem of core mathematics which is provable in second-order arithmetic and implies Π^1_2 comprehension.