Recursion Theory and Symbolic Dynamics

Stephen G. Simpson

Pennsylvania State University

http://www.math.psu.edu/simpson/

simpson@math.psu.edu

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Max Planck Institute for Mathematics

Bonn, Germany

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Abstract.

A set $P \subseteq \{0,1\}^{\mathbb{N}}$ may be viewed as a *mass problem*, i.e., a decision problem with more than one solution. By definition, the *solutions* of P are the elements of P. A mass problem is said to be *solvable* if at least one of its solutions is recursive. A mass problem P is said to be *Muchnik reducible* to a mass problem Q if for each solution of Q there exists a solution of P which is Turing reducible to the given solution of Q. A *Muchnik degree* is an equivalence class of mass problems under mutual Muchnik reducibility.

A set $P \subseteq \{0,1\}^{\mathbb{N}}$ is said to be Π^0_1 if it is *effectively closed*, i.e., it is the complement of the union of a recursive sequence of basic open sets. The lattice $\mathcal{P}_w$ of Muchnik degrees of mass problems associated with nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ has been investigated by the speaker and others. It is known that $\mathcal{P}_w$ contains many specific, natural Muchnik degrees which are related to various topics in the foundations of mathematics. Among these topics are algorithmic randomness, reverse mathematics, almost everywhere domination, hyperarithmeticity, resource-bounded computational complexity, Kolmogorov complexity, and subrecursive hierarchies.

Let A be a finite set of symbols. The *full two-dimensional shift* on A is the dynamical system consisting of the natural action of the group $\mathbb{Z} \times \mathbb{Z}$ on

Abstract, continued.

the compact space $A^{\mathbb{Z}\times\mathbb{Z}}$. A *two-dimensional subshift* is a nonempty closed subset of $A^{\mathbb{Z}\times\mathbb{Z}}$ which is invariant under the action of $\mathbb{Z}\times\mathbb{Z}$. A two-dimensional subshift is said to be *of finite type* if it is defined by a finite set of forbidden configurations. The two-dimensional subshifts of finite type are known to form an important class of dynamical systems, with connections to mathematical physics, etc.

Clearly every two-dimensional subshift of finite type is a nonempty Π^0_1 subset of $A^{\mathbb{Z}\times\mathbb{Z}}$, hence its Muchnik degree belongs to $\mathcal{P}_w$. Conversely, we prove that every Muchnik degree in $\mathcal{P}_w$ is the Muchnik degree of a two-dimensional subshift of finite type. The proof of this result uses tilings of the plane. We present an application of this result to symbolic dynamics. Our application is stated purely in terms of two-dimensional subshifts of finite type, with no mention of Muchnik degrees.

We begin with the 1-dimensional case.

A *dynamical system* consists of
a nonempty set X (the set of *states*)
plus a mapping $T : X \to X$
(the *state transition operator*).

Throughout this talk we assume that X is compact and metrizable. We also assume that T is continuous, one-to-one, and onto.

Example. Let A be a finite set of symbols. $A^{\mathbb{Z}}$ is the set of bi-infinite sequences of symbols from A. The *shift operator* $S : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is given by $S(x)(n) = x(n+1)$ for all $x \in A^{\mathbb{Z}}$.

The dynamical system consisting of the compact metrizable space $A^{\mathbb{Z}}$ and the shift operator S is known as the *full shift* on A.

Let X be a nonempty closed subset of $A^{\mathbb{Z}}$ which is *invariant under the shift operator*, i.e., $x \in X \iff S(x) \in X$ for all x.

The dynamical system consisting of the compact metrizable space X together with the shift operator S (actually $S \restriction X$) is known as a *subshift* on A.
It is a subsystem of the full shift on A.

There are many different kinds of subshifts.
Subshifts are very useful for describing the behavior of dynamical systems in general.

The study of subshifts for their own sake is called *symbolic dynamics*.

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Every subshift $X \subseteq A^{\mathbb{Z}}$ is defined by a set of *forbidden words*. Namely, for an appropriate set F of finite sequences of symbols from A,

$$X = \{x \in A^{\mathbb{Z}} \mid x \text{ contains no consecutive subsequence belonging to } F\}.$$

If F is finite, we say that X is *of finite type*.

Subshifts of finite type have been studied extensively. It is easy to see that every 1-dimensional subshift of finite type contains periodic points.

Note: F is recursive if and only if X is Π^0_1.

Cenzer/Dashti/King 2007 have constructed a 1-dimensional Π^0_1 subshift which contains no recursive points, hence no periodic points.

Many questions regarding 1-dimensional Π^0_1 subshifts remain open.

We now turn to the 2-dimensional case.

As before, let A be a finite set of symbols.

Let $A^{\mathbb{Z}\times\mathbb{Z}}$ be the set of doubly bi-infinite double sequences of symbols from A.

This is again a compact metrizable space.

Points of $A^{\mathbb{Z}\times\mathbb{Z}}$ may be viewed as *tilings of the plane*, in the sense of Wang 1961.

Tiling problems were studied by logicians during the years 1960–1980.

The connection with dynamical systems was noticed only relatively recently.

A *2-dimensional dynamical system* consists of a nonempty set X and a commuting pair of maps $T_1, T_2 : X \to X$. As before we assume X compact metrizable, T_1, T_2 continuous one-to-one onto.

The *full 2-dimensional shift* on A is the dynamical system consisting of $A^{\mathbb{Z}\times\mathbb{Z}}$ with shift operators $S_1, S_2 : A^{\mathbb{Z}\times\mathbb{Z}} \to A^{\mathbb{Z}\times\mathbb{Z}}$ given by
$S_1(x)(m,n) = x(m+1,n)$ and
$S_2(x)(m,n) = x(m,n+1)$.

A *2-dimensional subshift* on A is
a nonempty closed set $X \subseteq A^{\mathbb{Z}\times\mathbb{Z}}$
which is invariant under S_1 and S_2.

Note that (X, S_1, S_2) is again a 2-dimensional dynamical system. It is a subsystem of the full 2-dimensional shift on A.

As in the 1-dimensional case, every 2-dimensional subshift X is defined by a set F of forbidden configurations.

If F is finite, X is said to be *of finite type*.

Here, by a *configuration* we mean a "2-dimensional word," i.e., a member of $A^{\{1,\ldots,r\}\times\{1,\ldots,r\}}$ for some positive integer r.

2-dimensional subshifts of finite type are important in dynamical systems theory. An example is the Ising model in mathematical physics.

History:

Berger 1966 answered a question of Wang 1961 by constructing a 2-dimensional subshift of finite type with no periodic points.

Berger 1966 showed that it is undecidable whether a given finite set of forbidden configurations defines a (nonempty!) 2-dimensional subshift.

Myers 1974 constructed
a 2-dimensional subshift of finite type
with no recursive points.

Hochman/Meyerovitch 2007 proved:
a real number $h \geq 0$ is the entropy of
a 2-dimensional subshift of finite type if and only if h is *right recursively enumerable*.
This means that h is the limit of a recursive decreasing sequence of rational numbers.

Using the methods of Robinson 1971 and Myers 1974, I have proved:

Theorem 1 (Simpson 2007). The Medvedev degrees of 2-dimensional subshifts of finite type are the same as the Medvedev degrees of nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$.

This theorem is useful, because we can then apply known results from recursion theory to study 2-dimensional subshifts of finite type.

Below we shall present one such application.

Our application will be stated purely in terms of 2-dimensional subshifts of finite type, with no mention of Medvedev degrees and no mention of recursion theory.

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To state our application, we need some easy definitions which make perfect sense for all dynamical systems.

Definition. Let X and Y be 2-dimensional subshifts on k and l symbols respectively. The Cartesian product $X \times Y$ and the disjoint union $X + Y$ are 2-dimensional subshifts on kl and $k + l$ symbols respectively.

Definition. Let (X, S_1, S_2) be a 2-dimensional subshift on k symbols. Let a, b, c, d be integers with $ad - bc \neq 0$. Then, the system $(X, S_1^a S_2^b, S_1^c S_2^d)$ is canonically isomorphic to a 2-dimensional subshift on $k^{|ad-bc|}$ symbols.

Definition. If $\mathcal{U}$ is a set of 2-dimensional subshifts, let cl($\mathcal{U}$) be the closure of $\mathcal{U}$ under the above operations.

Definition. If X and Y are 2-dimensional subshifts, a *shift morphism* from X to Y is a continuous mapping $f : X \to Y$ which commutes with the shift operators.

In other words, $f(S_1(x)) = S_1(f(x))$ and $f(S_2(x)) = S_2(f(x))$ for all $x \in X$.

Now for the application.

Theorem 2 (Simpson 2007).
There is an infinite set $\mathcal{W}$ of 2-dimensional subshifts of finite type, such that for any partition $\mathcal{U}, \mathcal{V}$ of $\mathcal{W}$, and for any $X \in \mathrm{cl}(\mathcal{U})$ and $Y \in \mathrm{cl}(\mathcal{V})$, there is no shift morphism from X to Y or vice versa.

Theorem 2 follows from Theorem 1 plus a previously known recursion-theoretic result:

> There is an infinite set of nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ whose Medvedev degrees are independent.

This known recursion-theoretic result is proved by means of a priority argument.

We now discuss some ingredients of Theorems 1 and 2 and their proofs.

A subset of $A^{\mathbb{Z}\times\mathbb{Z}}$ or of $A^{\mathbb{Z}}$ or of $\{0,1\}^{\mathbb{N}}$ is *effectively closed* if it is the complement of the union of a recursive sequence of basic open sets. Here a *basic open set* is any set of the form $N_\sigma = \{x \mid x \upharpoonright \mathrm{dom}(\sigma) = \sigma\}$ where σ is a finite partial function.

By definition, a set is Π^0_1 if and only if it is effectively closed.

The spaces $A^{\mathbb{Z}\times\mathbb{Z}}$ and $A^{\mathbb{Z}}$ and $\{0,1\}^{\mathbb{N}}$ are recursively homeomorphic to each other.

Hence, Π^0_1 sets in any of them are recursively homeomorphic to Π^0_1 sets in all of them.

Clearly subshifts of finite type are Π^0_1. More generally, any subshift defined by a recursive sequence of forbidden configurations is Π^0_1.

Let X and Y be Π^0_1 sets.

Y is *Medvedev reducible to* X if there exists a partial recursive functional from X to Y.

X and Y are *Medvedev equivalent* if each is Medvedev reducible to the other.

Recursively homeomorphic sets are Medvedev equivalent, but not conversely.

A *Medvedev degree* is an equivalence class under Medvedev equivalence.

The Medvedev degrees of all nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ under Medvedev reducibility form a distributive lattice, denoted $\mathcal{P}_s$.

It is known that $\mathcal{P}_s$ is structurally rich.

Theorem 1 says that the Medvedev degrees of <u>2-dimensional</u> subshifts of finite type are precisely the Medvedev degrees in $\mathcal{P}_s$.

By contrast, all <u>1-dimensional</u> subshifts of finite type are of Medvedev degree $\mathbf{0}$.

Thus, the 2-dimensional case is much more complicated than the 1-dimensional case.

Let X and Y be 2-dimensional subshifts.

A basic fact concerning shift morphisms:

Each shift morphism $f : X \to Y$ is describable in a very simple manner as a *block code*. This means that $f(x)(m,n)$ depends only on $x(m \pm i, n \pm j)$, $i,j \in \{0,\ldots,r\}$ for some fixed r.

In particular, each shift morphism is given by a recursive functional. Thus, the existence of a shift morphism from X to Y implies that Y is Medvedev reducible to X.

Define $X \geq Y$ if there exists a shift morphism from X to Y. Define $X \equiv Y$ if $X \geq Y$ and $Y \geq X$. The $\equiv$-equivalence classes form a distributive lattice. We have:

Theorem 3 (Simpson 2007). There is a canonical lattice homomorphism of the lattice of $\equiv$-equivalence classes of 2-dimensional subshifts of finite type, onto the lattice $\mathcal{P}_s$.

In all of these lattices, the supremum and infimum are given by $X \times Y$ and $X + Y$.

If X is a 2-dimensional subshift of finite type, there are surely some interesting relationships between the dynamical properties of X and the Medvedev degree of X.

These relationships remain to be explored.

Of even greater potential interest is the Muchnik degree of X.

Definition. Y is *Muchnik reducible to* X if each point of X is carried to some point of Y by some recursive functional. X and Y are *Muchnik equivalent* if each is Muchnik reducible to the other. A *Muchnik degree* is an equivalence class under Muchnik equivalence. The Muchnik degrees of nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ form a distributive lattice, denoted $\mathcal{P}_w$.

Trivially Medvedev reducibility implies Muchnik reducibility. Thus we have a lattice homomorphism of $\mathcal{P}_s$ onto $\mathcal{P}_w$. Hence, Theorems 1 and 3 hold for $\mathcal{P}_w$.

Moreover $\mathcal{P}_w$, like $\mathcal{P}_s$, is structurally rich.

The advantage of $\mathcal{P}_w$ over $\mathcal{P}_s$ is:
$\mathcal{P}_w$ contains many specific, natural degrees which are motivated by the idea of *mass problems* in recursion theory.

These specific, natural degrees in $\mathcal{P}_w$ are linked to foundational topics:

- algorithmic randomness
- reverse mathematics
- almost everywhere domination
- diagonal nonrecursiveness
- hyperarithmeticity
- resource-bounded complexity
- Kolmogorov complexity
- subrecursive hierarchies

Some examples of Muchnik degrees in $\mathcal{P}_w$:

$\mathbf{0}$ = the bottom degree in $\mathcal{P}_w$
= the Muchnik degree of $\{x \mid x$ is recursive$\}$

$\mathbf{1}$ = the top degree in $\mathcal{P}_w$ = the Muchnik degree of $\{x \mid x$ is a completion of Peano Arithmetic$\}$

$\mathbf{r}_1$ = Muchnik degree of $\{x \in \{0,1\}^{\mathbb{N}} \mid x$ is *random*$\}$ (in the sense of Martin-Löf)

$\mathbf{r}_2$ = Muchnik degree of $\{x \in \{0,1\}^{\mathbb{N}} \mid x$ is random relative to $0'$, the Halting Problem$\}$

$\mathbf{d}$ = the Muchnik degree of $\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is *diagonally nonrecursive*$\}$ (i.e., $f(n) \not\simeq \varphi_n^{(1)}(n)$ for all n)

$\mathbf{d}_{\mathsf{REC}}$ = the Muchnik degree of $\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is diagonally nonrecursive and *recursively bounded*$\}$ (i.e., f is bounded by a recursive function)

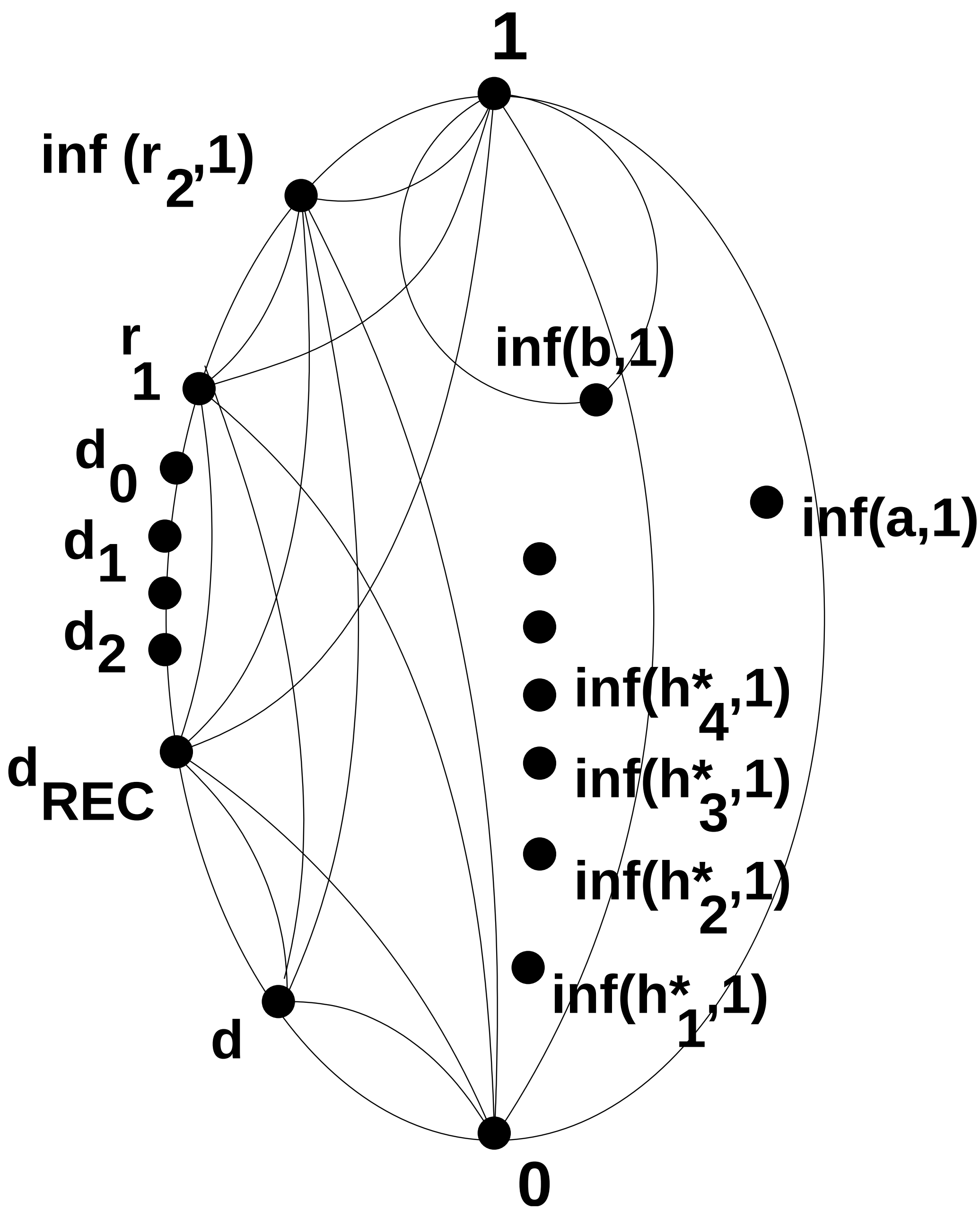
$\mathbf{d}_\alpha$ = the Muchnik degree of $\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is diagonally nonrecursive and α-*recursively bounded*$\}$ (bounded at level α of the Wainer hierarchy), $\alpha \le \varepsilon_0$

$\mathbf{a}$ = the Muchnik degree of a recursively enumerable set

$\mathbf{h}_\alpha$ = the Muchnik degree of $0^{(\alpha)}$, $\alpha < \omega_1^{\mathsf{CK}}$

$\mathbf{h}_\alpha^*$ = the blurred version of $\mathbf{h}_\alpha$, $\alpha < \omega_1^{\mathsf{CK}}$

$\mathbf{b}$ = the Muchnik degree of $\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is almost everywhere dominating$\}$



A picture of $\mathcal{P}_w$. Here $\mathbf{a}$ is any r.e. degree, $\mathbf{h}$ = hyperarithmeticity, $\mathbf{r}$ = randomness, $\mathbf{b}$ = almost everywhere domination, $\mathbf{d}$ = diagonal nonrecursiveness.

By Theorem 1, each of the black dots in the above picture is the Muchnik degree of a 2-dimensional subshift of finite type.

Thus we have apparently uncovered some interesting classes of 2-dimensional subshifts of finite type.

A basic result concerning $\mathcal{P}_w$ is as follows:

Embedding Lemma (Simpson 2004).
Let $\mathbf{s}$ be the Muchnik degree of a Σ^0_3 set. Then $\inf(\mathbf{s},\mathbf{1})$ belongs to $\mathcal{P}_w$.

Combining this with Theorem 1, we obtain:

Theorem 4 (Simpson 2007). Let $\mathbf{s}$ be the Muchnik degree of a Σ^0_3 set. Then there exists a 2-dimensional subshift of finite type whose Muchnik degree is $\inf(\mathbf{s},\mathbf{1})$.

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Some of my papers are available at

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