Recursion Theory and Symbolic Dynamics

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Abstract

A set $P \subseteq \{0,1\}^{\mathbb{N}}$ may be viewed as a *mass problem*, i.e., a decision problem with more than one solution. By definition, the *solutions* of P are the elements of P. A mass problem is said to be *solvable* if at least one of its solutions is recursive. A mass problem P is said to be *Muchnik reducible* to a mass problem Q if for each solution of Q there exists a solution of P which is Turing reducible to the given solution of Q. A *Muchnik degree* is an equivalence class of mass problems under mutual Muchnik reducibility.

A set $P \subseteq \{0,1\}^{\mathbb{N}}$ is said to be Π^0_1 if it is *effectively closed*, i.e., it is the complement of the union of a recursive sequence of basic open sets. The lattice $\mathcal{P}_w$ of Muchnik degrees of mass problems associated with nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$ has been investigated by the speaker and others. It is known that $\mathcal{P}_w$ contains many specific, natural Muchnik degrees which are related to various topics in the foundations of mathematics. Among these topics are algorithmic randomness, reverse mathematics, almost everywhere domination, hyperarithmeticity, resource-bounded computational complexity, Kolmogorov complexity, and subrecursive hierarchies.

Let A be a finite set of symbols. The *full two-dimensional shift* on A is the dynamical system consisting of the natural action of the group $\mathbb{Z} \times \mathbb{Z}$ on the compact space $A^{\mathbb{Z} \times \mathbb{Z}}$. A *two-dimensional subshift* is a nonempty closed subset of $A^{\mathbb{Z} \times \mathbb{Z}}$ which is invariant under the action of $\mathbb{Z} \times \mathbb{Z}$. A two-dimensional subshift is said to be *of finite type* if it is defined by a finite set of forbidden configurations. The two-dimensional subshifts of finite type are known to form an important class of dynamical systems, with connections to mathematical physics, etc.

Clearly every two-dimensional subshift of finite type is a nonempty Π^0_1 subset of $A^{\mathbb{Z} \times \mathbb{Z}}$, hence its Muchnik degree belongs to $\mathcal{P}_w$. Conversely, we prove that every Muchnik degree in $\mathcal{P}_w$ is the Muchnik degree of a two-dimensional subshift of finite type. The proof of this result uses tilings of the plane. We present an application of this result to symbolic dynamics. Our application is stated purely in terms of two-dimensional subshifts of finite type, with no mention of Muchnik degrees.