Mass problems associated with effectively closed sets

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Abstract:

We begin with a brief introduction to mass problems in general. After that, the purpose of the talk is to introduce $\mathcal{P}_w$, the lattice of Muchnik degrees of mass problems associated with nonempty effectively closed sets in the Cantor space. We show that $\mathcal{P}_w$ is a countable distributive lattice with 0 and 1. We show that the top element of $\mathcal{P}_w$ is the Muchnik degree of the problem of finding a complete consistent theory which extends Peano Arithmetic. The G"odel Incompleteness Theorem tells us that this problem is unsolvable. Instead of Peano Arithmetic we could take any theory which, like Peano Arithmetic, is recursively axiomatizable and effectively essentially undecidable. It turns out that the effectively closed sets associated with all such theories are not only Muchnik equivalent but also recursively homeomorphic to each other. As time permits we shall exhibit some other interesting examples of specific, natural Muchnik degrees in $\mathcal{P}_w$.

The simplest proof that $\mathcal{P}_w$ has a top element is based on the following considerations.

For functions $f \in \mathbb{N}^{\mathbb{N}}$ define $(f)_i(j) = f(3^i 5^j)$. Instead of $3^i 5^j$ we could use any recursive pairing function on $\mathbb{N}$. Note that $(f)_i \in \mathbb{N}^{\mathbb{N}}$ and f encodes the sequence $(f)_i$, $i = 0, 1, 2, \ldots$.

For strings $\tau \in \mathbb{N}^{<\mathbb{N}}$ define $(\tau)_i(j) \simeq \tau(3^i 5^j)$. Note that $(\tau)_i \in \mathbb{N}^{<\mathbb{N}}$ and $|(\tau)_i| \leq |\tau|$.

If T_i, $i = 0, 1, 2, \ldots$ is a sequence of trees, define

$$T = \prod_{i=0}^{\infty} T_i = \{\tau \mid (\forall i \leq |\tau|)\,((\tau)_i \in T_i)\} \ .$$

Note that (1) T is a tree, (2) if the T_i for $i = 0, 1, 2, \ldots$ are uniformly recursive, then T is recursive, (3) f is a path through T if and only if $\forall i\,((f)_i$ is a path through $T_i)$.

To prove that $\mathcal{P}_w$ has a top element.

Define $P_e = \{X \in 2^{\mathbb{N}} \mid \varphi_e^{(1),X}(0)\uparrow\}$. Note that P_e, $e = 0, 1, 2, \ldots$ is our standard recursive enumeration of the Π^0_1 subsets of $2^{\mathbb{N}}$. Define $T_e = \{\tau \in 2^{<\mathbb{N}} \mid \varphi_{e,|\tau|}^{(1),\tau}(0)\uparrow\}$. Note that T_e for $e = 0, 1, 2, \ldots$ is a uniformly recursive sequence of trees, and $P_e = \{\text{paths through } T_e\}$.

The idea now is to define

$$T = 2^{<\mathbb{N}} \cap \prod_{e=0}^{\infty} T_e$$

and $P = \{\text{paths through } T\}$. Then P is a Π^0_1 subset of $2^{\mathbb{N}}$ and

$$P = \prod_{e=0}^{\infty} P_e = \{X \in 2^{\mathbb{N}} \mid \forall e\,((X)_e \in P_e)\}$$

so for all $X \in P$ we have $(X)_e \in P_e$ for all e. Thus $P_e \leq_w P$ for all e.

The only difficulty here is that $P = \emptyset$, because $P_e = \emptyset$ for some e's.

We now repair the construction.

By König's Lemma, $P_e = \emptyset$ if and only if T_e is finite. Define

$$h(T_e) = \text{the height of } T_e = \sup\{|\tau| \mid \tau \in T_e\}$$

and

$$T_e^+ = T_e \cup \{\tau \in 2^{<\mathbb{N}} \mid h(T_e) \leq |\tau| \wedge \tau \restriction h(T_e) \in T_e\}$$

and

$$P_e^+ = \{\text{paths through } T_e^+\}.$$

Note that T_e^+ is a uniformly recursive sequence of infinite recursive trees, and $T_e^+ = T_e$ whenever T_e is infinite. Hence the sets P_e^+, $e = 0, 1, 2, \ldots$ are uniformly Π^0_1 and nonempty, and $P_e^+ = P_e$ whenever P_e is nonempty.

We now define

$$T^+ = 2^{<\mathbb{N}} \cap \prod_{e=0}^{\infty} T_e^+$$

and

$$P^+ = \prod_{e=0}^{\infty} P_e^+ = \{\text{paths through } T^+\} \, .$$

Then P^+ is a nonempty Π^0_1 subset of $2^{\mathbb{N}}$. Moreover, for all $X \in P$ and all e we have $(X)_e \in P_e^+$, hence $(X)_e \in P_e$ whenever P_e is nonempty.

Thus P^+ represents the top element of $\mathcal{P}_w$.

This completes the proof that $\mathcal{P}_w$ has a top element.

Let L be a *language*, i.e., a set of predicates. We usually assume that L is finite, or at worst recursive. We consider the first-order predicate calculus over L.

A *theory* is an ordered pair (L, S) where L is a language and S is a set of L-sentences which is consistent and closed under logical consequence, i.e., $S \vdash F$ if and only if $F \in S$. We usually write S instead of (L, S).

The theory S is said to be *complete* if for all L-sentences F either $F \in S$ or $\neg F \in S$.

A *completion of* S is a complete L-theory extending S.

Lindenbaum's Lemma says that every theory has at least one completion. Hence, S is complete if and only if S has exactly one completion.

The theory S is said to be *decidable* if the set of Gödel numbers $\#(S) = \{\#(F) \mid F \in S\}$ is recursive.

The theory S is said to be *recursively axiomatizable* if S is the closure under logical consequence of a recursive set of L-sentences, called the *axioms* of S.

Easy facts:

1. S is recursively axiomatizable if and only if S is recursively enumerable.

2. If S is decidable, then S is recursively axiomatizable.

3. If S is complete and recursively axiomatizable, then S is decidable.

4. If S is decidable, then S has a decidable completion.

A theory S is said to be *essentially undecidable* if there is no decidable L-theory extending S. Equivalently, S has no decidable completion.

Peano Arithmetic or PA is the theory in the language of arithmetic $\{+, \times, 0, 1, =\}$ containing the basic axioms plus the induction scheme.

By Tarski/Mostowski/Robinson, PA is essentially undecidable.

(In fact, Q is essentially undecidable. Note that Q is a very weak, finitely axiomatizable subtheory of PA.)

It follows that every recursively axiomatizable theory extending PA (or even, extending Q) is incomplete. This is the Gödel/Rosser Incompleteness Theorem.

Given a recursively axiomatizable theory S, let $C(S)$ be the set of characteristic functions of sets of the form $\#(S_1)$ where S_1 is a completion of S.

Note that $C(S)$ is a nonempty Π^0_1 subset of $2^{\mathbb{N}}$.

There is a converse theorem due to Hanf and Peretyatkin:

> Given a nonempty Π^0_1 set $P \subseteq 2^{\mathbb{N}}$, we can find a <u>finitely axiomatizable</u> theory S such that $C(S)$ is recursively homeomorphic to P.

Hence, $C(S)$ and P have the same Muchnik degree.

We wish to show that the Muchnik degree of $C(\text{PA})$ is $\mathbf{1}$, the top degree in $\mathcal{P}_w$.

Let S be a completion of PA.

We say that $X \in 2^{\mathbb{N}}$ is *representable* in S if there exists a formula $F(x)$ with one free variable such that for all n, $X(n) = 1$ if $S \vdash F(\underline{n})$, and $X(n) = 0$ if $S \vdash \neg F(\underline{n})$.
Since S is complete, it follows that $X \leq_T \#(S)$.

We shall show that for every nonempty Π^0_1 set $P \subseteq 2^{\mathbb{N}}$ there exists $X \in P$ such that X is representable in S.

The idea is that $P = P_e = \{\text{paths through } T_e\}$ where T_e is a recursive subtree of $2^{<\mathbb{N}}$. Let T_e^* be T_e according to S. We define $X = X_e^* \restriction \mathbb{N}$ where X_e^* is the leftmost path through T_e^* according to S.

The difficulty is that, even though T_e is infinite, we may have $S \vdash$ "T_e is finite", hence $S \vdash$ "T_e has no paths", so X_e^* does not make sense.

The solution is to use T_e^+ instead of T_e. Clearly PA $\vdash$ "T_e^+ is infinite", hence $S \vdash$ "T_e^+ is infinite", hence we can let X_e^* be the leftmost path in $(T_e^+)^*$. We need to show that $X = X_e^* \restriction \mathbb{N}$ is actually a path through T_e. This holds because $T_e = (T_e^+)^* \cap 2^{<\mathbb{N}}$.

To make this clearer, consider $M(S)$, the *term model* of S. To define $M(S)$ we consider μ-*terms*, i.e., terms of the form

$$t \;=\; \mu x\,(F(x) \vee \neg\,\exists x\,F(x))$$

where μ is the least number operator and $F(x)$ is a formula with one free variable. Two μ-terms t_1 and t_2 are said to be *equivalent* if $S \vdash t_1 = t_2$. The elements of $M(S)$ are the equivalence classes of μ-terms.

For all sentences F, we have $S \vdash F$ if and only if $M(S) \models F$. In other words, $M(S)$ is a model of S. Moreover, $M(S)$ is the smallest model of S. In model-theoretic terms, $M(S)$ is what is called the *prime model* of S.

For example, if S is true arithmetic, then $M(S)$ is just $\mathbb{N}$, the standard model of arithmetic. In general, $M(S)$ is a nonstandard model of arithmetic.

We identify $n \in \mathbb{N}$ with the equivalence class of the μ-term

$$\mu x \, (x = \underline{n} \vee \neg \, \exists x \, (x = \underline{n}))$$

where $\underline{n} = \underbrace{1 + \cdots + 1}_{n}$. Thus $\mathbb{N}$ is a submodel of $M(S)$ and an initial segment of $M(S)$.

Now, as above, let $e \in \mathbb{N}$ be such that T_e is an infinite recursive subtree of $2^{<\mathbb{N}}$. The difficulty is that $M(S)$ may think T_e is finite, i.e., we may have $M(S) \models$ "T_e is finite".

The solution to this difficulty is to consider T_e^+ instead of T_e. Clearly $M(S) \models$ "T_e^+ is infinite", so let $(T_e^+)^*$ be T_e^+ as understood in $M(S)$.

Note that T_e is an initial segment of $(T_e^+)^*$, namely $T_e = (T_e^+)^* \cap 2^{<\mathbb{N}}$.

Let X_e^* be the leftmost path through $(T_e^+)^*$ as understood in $M(S)$. Then $X_e^* \upharpoonright \mathbb{N}$ is a path through T_e. Moreover, $X_e^* \upharpoonright \mathbb{N}$ is representable in S. Thus we have a path through T_e which is representable in S. This completes the proof.

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