

Two Topics in the Theory of Computability and Unsolvability

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Topic 1:

An Extension of the R.E. Turing Degrees

Topic 2:

Almost Everywhere Domination

(joint work with Natasha Dobrinen)

Background: the r.e. Turing degrees

For $X, Y \subseteq \omega = \{0, 1, 2, \dots\}$,

X is *Turing reducible* to Y (i.e., $X \leq_T Y$)

iff X is computable using an oracle for Y .

The *Turing degrees* are the equivalence classes under \leq_T , ordered by \leq_T .

The l.u.b. of two Turing degrees is given by

$$X \oplus Y = \{2n \mid n \in X\} \cup \{2n + 1 \mid n \in Y\}.$$

$X \subseteq \omega$ is *r.e.* (i.e., *recursively enumerable*)

iff it is the range of a recursive function.

An *r.e. Turing degree* is a Turing degree that contains an r.e. set.

\mathcal{R}_T is the semilattice of r.e. Turing degrees.

This structure has been studied extensively by recursion theorists.

Background, continued:

\mathcal{R}_T is the semilattice of r.e. Turing degrees.

Intensive study of lattice-theoretic properties of \mathcal{R}_T has yielded nothing for f.o.m.

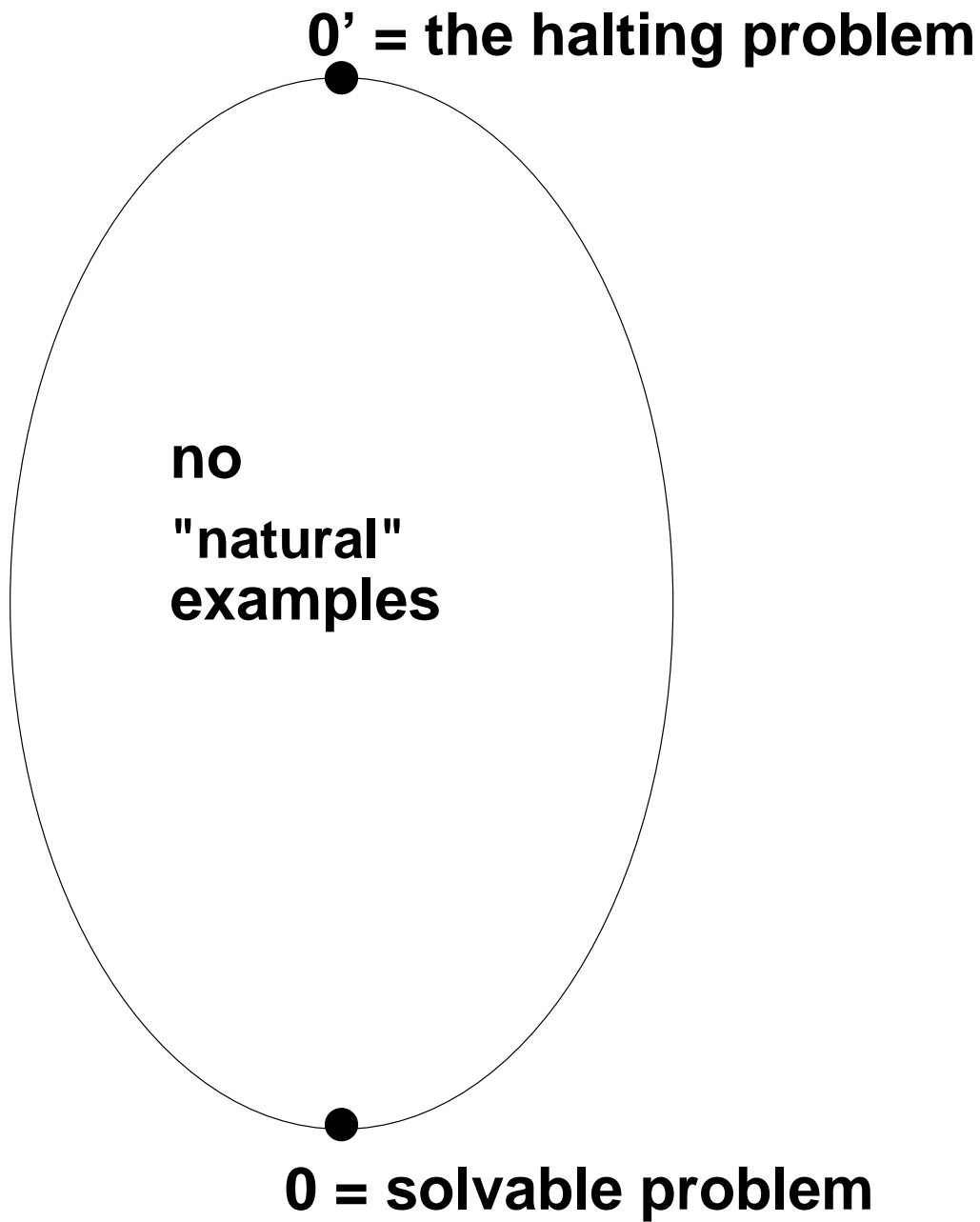
Moreover, after 50 years, the only known specific examples of r.e. Turing degrees are the bottom and top elements of \mathcal{R}_T .

$\mathbf{0}$ = Turing degree of recursive sets.

$\mathbf{0}'$ = Turing degree of the Halting Problem.

There are infinitely many r.e. Turing degrees, but there are no known “natural” ones, other than $\mathbf{0}$ and $\mathbf{0}'$.

A picture of the r.e. Turing degrees, \mathcal{R}_T :



The main point:

We embed the upper semilattice of r.e. Turing degrees, \mathcal{R}_T , into another structure, \mathcal{P}_w , which is

1. slightly larger,
2. somewhat better behaved,
3. much more relevant to f.o.m.
(= foundations of mathematics).

What is this wonderful structure \mathcal{P}_w ?

Briefly, \mathcal{P}_w is the lattice of Muchnik degrees of nonempty Π_1^0 subsets of 2^ω .

An extension of the r.e. Turing degrees:

We define the Muchnik lattice \mathcal{P}_w .

The Cantor space is $2^\omega = \{X : \omega \rightarrow \{0, 1\}\}$.

(Note: The compactness of 2^ω is relevant.)

For $P, Q \subseteq 2^\omega$, P is *Muchnik reducible* to Q ($P \leq_w Q$) iff every member of Q computes a member of P , i.e., $\forall Y \in Q \exists X \in P X \leq_T Y$.

Muchnik degrees are equivalence classes of subsets of 2^ω under \leq_w , ordered by \leq_w .

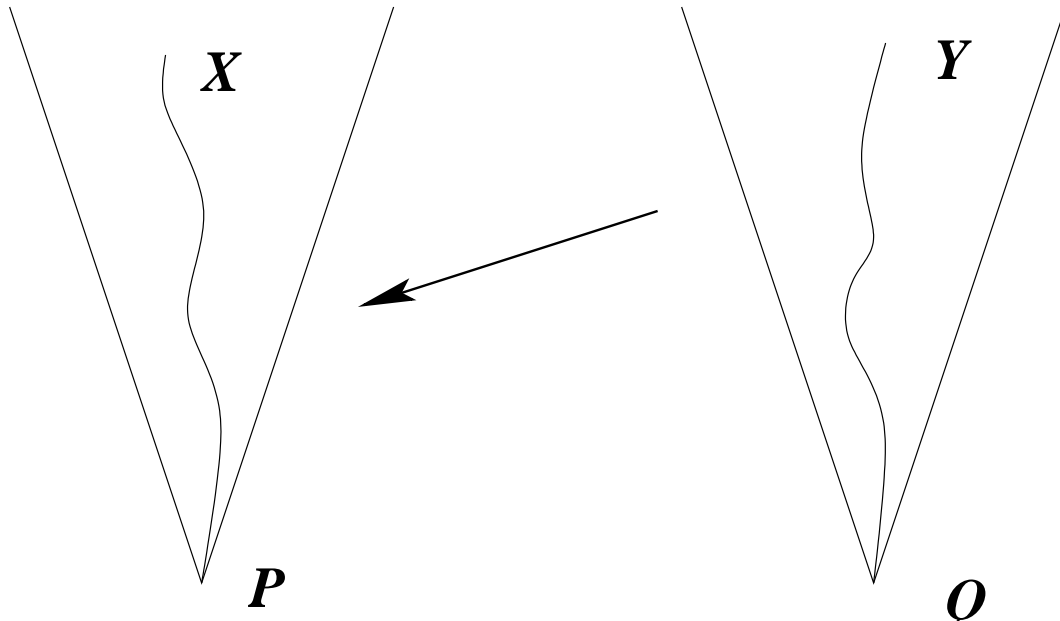
The l.u.b. of two Muchnik degrees is given by $P \times Q = \{X \oplus Y \mid X \in P \text{ and } Y \in Q\}$.

The g.l.b. is given by $P \cup Q$.

$P \subseteq 2^\omega$ is Π_1^0 iff P is the set of paths through a recursive subtree of the full binary tree of finite sequences of 0's and 1's.

\mathcal{P}_w is defined to be the lattice of Muchnik degrees of nonempty Π_1^0 subsets of 2^ω .

Muchnik reducibility:



$P \leq_w Q$ means:

$$\forall Y \in Q \exists X \in P X \leq_T Y.$$

P, Q are given by recursive subtrees of the full binary tree of finite sequences of 0's and 1's.

X, Y are infinite (nonrecursive) paths through P, Q respectively.

An extension, continued:

Properties of \mathcal{P}_w , the lattice of Muchnik degrees of nonempty Π_1^0 subsets of 2^ω :

(a) \mathcal{P}_w is a distributive lattice. Thus, its structure is more regular than that of \mathcal{R}_T .

\mathcal{P}_w has a bottom element and a top element:

$0 =$ the Muchnik degree of 2^ω ,

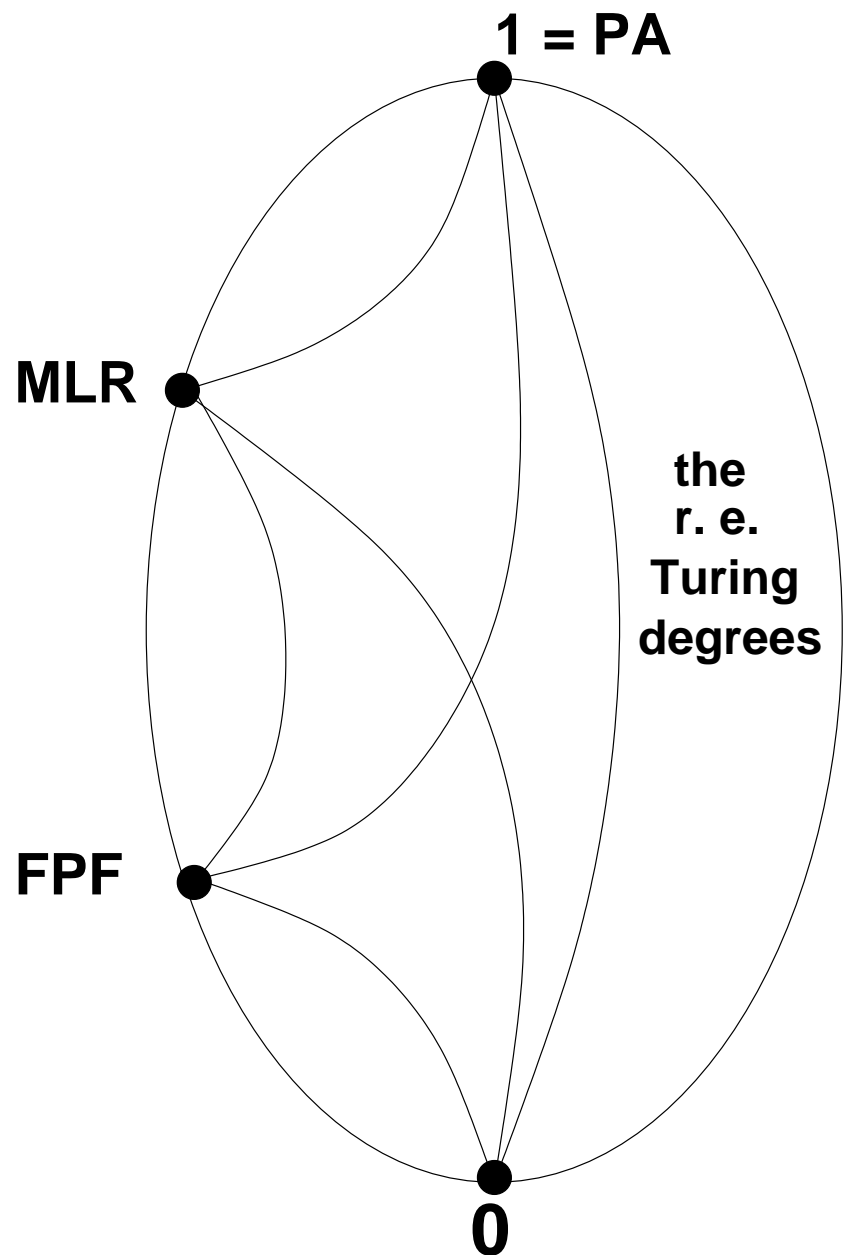
$1 = \text{PA} =$ the Muchnik degree of the set of completions of theories that are *sufficiently strong*, in the sense of the Gödel/Rosser Theorem: EFA, PA, Z_2 , ZFC,

(b) There are at least two other “natural” Muchnik degrees in \mathcal{P}_w . See below.

In these important senses, the Muchnik lattice \mathcal{P}_w is better than \mathcal{R}_T , the semilattice of r.e. Turing degrees. It overcomes some of the well known deficiencies of \mathcal{R}_T .

(Simpson, August 1999, on FOM)

A picture of the Muchnik lattice \mathcal{P}_w :



An extension, continued:

Two “natural” Muchnik degrees in \mathcal{P}_w .

MLR = the Muchnik degree of the set of Martin-Löf random sequences of 0's and 1's.
(essentially due to Kučera 1985)

FPF = the Muchnik degree of the set of fixed-point-free functions, in the sense of the Arslanov Completeness Criterion.
(Simpson 2002)

In \mathcal{P}_w we have $0 < \text{FPF} < \text{MLR} < \text{PA} = 1$.

F.o.m. connection: The Muchnik degrees MLR and FPF correspond to subsystems of WKL_0 which arise in the reverse mathematics of measure theory (Yu/Simpson 1990) and continuous functions (Giusto/Simpson 2000), respectively. The Muchnik degree PA corresponds to WKL_0 itself.

Problem: Find additional “natural” Muchnik degrees in \mathcal{P}_w .

MLR = the Muchnik degree of the set of
Martin-Löf random (1-random) reals
= the maximum Muchnik degree of a
 Π_1^0 subset of 2^ω of positive measure.

(implicit in Kučera 1985)

FPF = the Muchnik degree of the set of
fixed-point-free functions
= the Muchnik degree of the set of
diagonally non-recursive functions
= the Muchnik degree of the set of
effectively immune sets
= the Muchnik degree of the set of
effectively biimmune sets

(implicit in Jockusch 1989)

Some additional “natural” Muchnik degrees in \mathcal{P}_w :

MLR_2 = the Muchnik degree of $R_2 \cup \text{PA}$,
where R_2 is the set of 2-random reals,
and PA is the set of completions
of Peano Arithmetic.

FPF^n , $n = 2, 3, \dots$, where $\text{FPF}^1 = \text{FPF}$ and

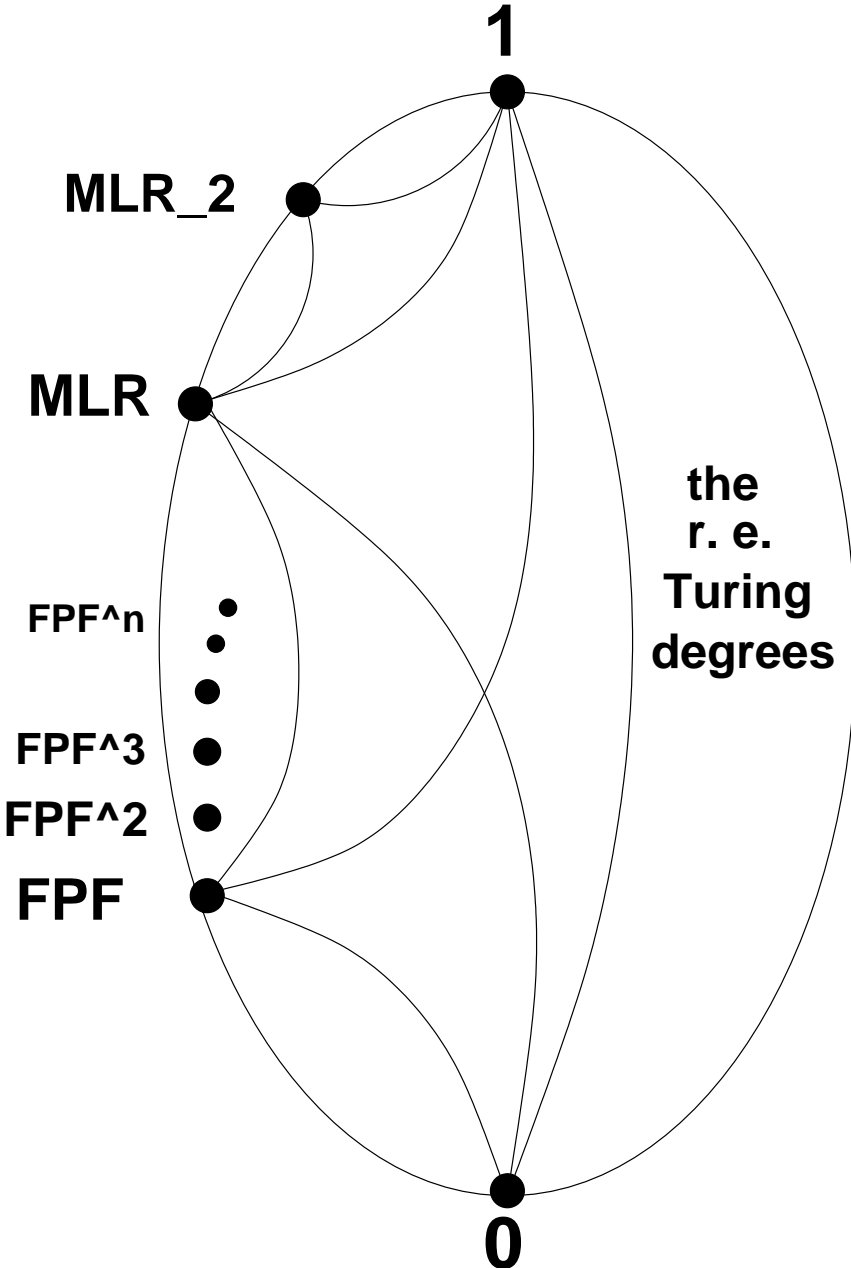
$$\text{FPF}^{n+1} = \{f \oplus g \mid f \in \text{FPF}^n, g \in \text{FPF}^f\}.$$

In \mathcal{P}_w we have:

$$\text{MLR} < \text{MLR}_2 < 1, \text{ and}$$

$$\text{FPF} < \text{FPF}^2 < \text{FPF}^3 < \dots < \text{FPF}^n < \dots$$

A picture of the Muchnik lattice \mathcal{P}_w :



Further properties of the Muchnik lattice \mathcal{P}_w .

1. \mathcal{P}_w is a countable distributive lattice.
Every countable distributive lattice is lattice embeddable in every initial segment of \mathcal{P}_w .
(Binns/Simpson 2001)

2. For all $P > 0$ there exist $P_1, P_2 < P$ such that $P = \text{l.u.b.}(P_1, P_2)$.
(Stephen Binns, 2002)

3. There does not exist $P < 1$ such that $\text{l.u.b.}(P, \text{MLR}) = 1$. (Simpson 2001)

4. There do not exist $P_1, P_2 > \text{MLR}$ such that $\text{g.l.b.}(P_1, P_2) = \text{MLR}$. (Simpson 2001)

5. If $P > 0$ is thin, then P is Muchnik incomparable with MLR . (Simpson 2001)

6. There do not exist $P_1, P_2 > 0$ such that $\text{g.l.b.}(P_1, P_2) = 0$. (trivial)

Embedding Theorem:

If $S \subseteq 2^\omega$ is Σ_3^0 , then for all nonempty Π_1^0 $P \subseteq 2^\omega$, there exists Π_1^0 $Q \subseteq 2^\omega$ such that $S \cup P \equiv_w Q$. (Simpson 2002)

Corollary:

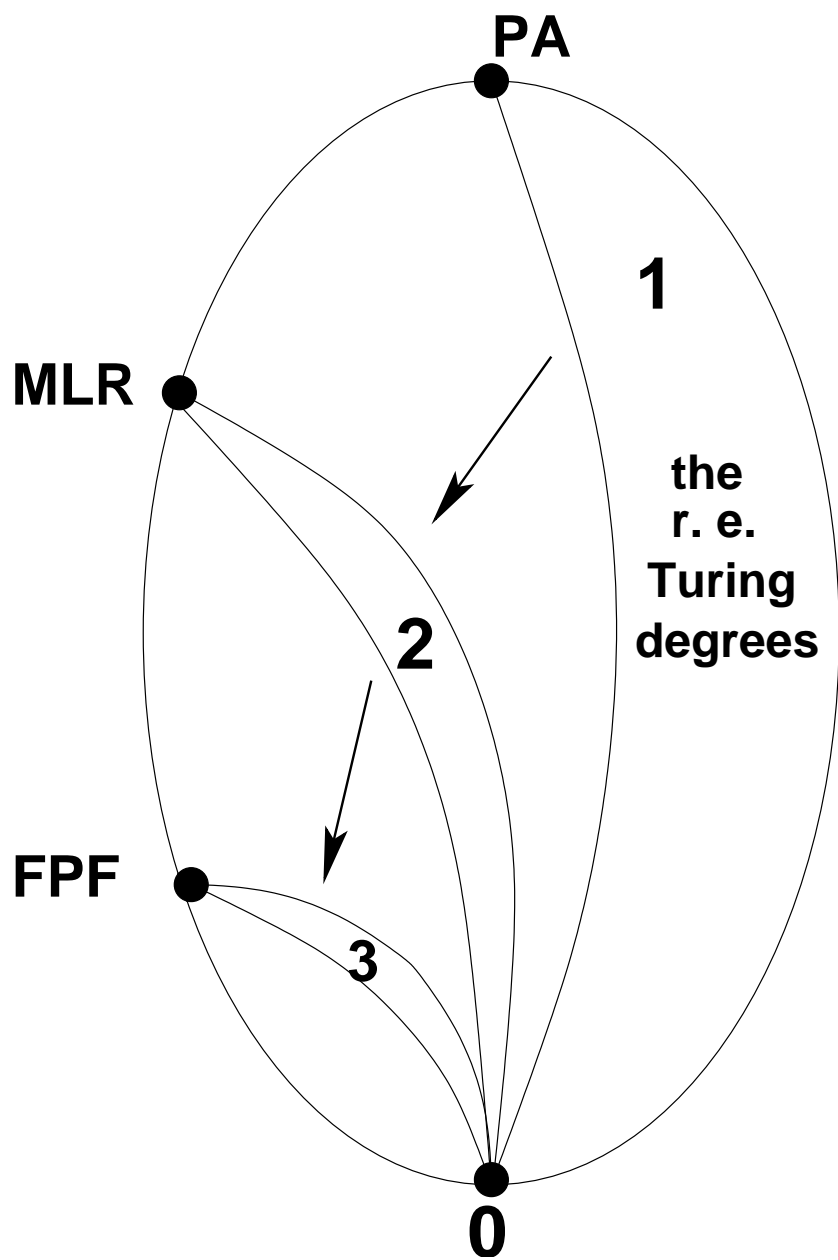
(embedding the r.e. Turing degrees)

Given a nonempty Π_1^0 $P \geq_w$ FPF, we have a semilattice embedding of the r.e. Turing degrees into \mathcal{P}_w , given by $X \mapsto \{X\} \cup P$. This embedding takes $\mathbf{0}$ to 0 , and $\mathbf{0}'$ to P . (Simpson 2002)

In particular, we have at least three “natural” embeddings of \mathcal{R}_T into \mathcal{P}_w :

1. Gödel/Rosser embedding, $X \mapsto \{X\} \cup \text{PA}$.
2. Martin-Löf embedding, $X \mapsto \{X\} \cup \text{MLR}$.
3. Arslanov embedding, $X \mapsto \{X\} \cup \text{FPF}$.

Three “natural” embeddings
of the r.e. Turing degrees
into the Muchnik lattice \mathcal{P}_w :



An extension, continued:

We have seen that the r.e. Turing degrees are embedded in \mathcal{P}_w .

Technical Note: Using a generalized Arslanov criterion, we can embed a wider class of Turing degrees: those that are $\leq 0'$ and n -REA for some $n \in \omega$.

Summary: The intensively studied semilattice of r.e. Turing degrees, \mathcal{R}_T , is included in the mathematically more natural, but less studied, Muchnik lattice, \mathcal{P}_w .

Moral: By studying the Muchnik lattice \mathcal{P}_w instead of the r.e. Turing degrees, recursion theorists could connect better to f.o.m., via reverse mathematics.

Two books on reverse mathematics:

1. Stephen G. Simpson

Subsystems of Second Order Arithmetic

Perspectives in Mathematical Logic

Springer-Verlag, 1999

XIV + 445 pages

To be reprinted in the ASL book series,
Perspectives in Logic

<http://www.math.psu.edu/simpson/sosoa/>

2. Stephen G. Simpson, editor

Reverse Mathematics 2001

(a volume of papers by various authors)

approximately 400 pages

To appear in the ASL book series,
Lecture Notes in Logic, 2004

<http://www.math.psu.edu/simpson/revmath/>

Topic 2:

Almost Everywhere Domination

(joint work with Natasha Dobrinen)

Basic definitions:

$$\omega = \{0, 1, 2, \dots\}.$$

$$2^\omega = \{X : \omega \rightarrow \{0, 1\}\}.$$

$$\mu(\{X \in 2^\omega \mid X(n) = i\}) = 1/2,$$

the “fair coin” probability measure on 2^ω .

$$\omega^\omega = \{f : \omega \rightarrow \omega\}.$$

f dominates g if $\exists m \forall n (n \geq m \Rightarrow f(n) > g(n))$.

Motivating result from set theory:

Let M be a countable transitive model of ZFC. Then for almost all $X \in 2^\omega$ (with respect to the fair coin measure on 2^ω), every $g \in M[X] \cap \omega^\omega$ is dominated by some $f \in M \cap \omega^\omega$.

This follows from the fact that measure algebras are *weakly* (ω, ω) -*distributive*:

$$\bigwedge_{n \in \omega} \bigvee_{k \in \omega} a_{nk} = \bigvee_{f \in \omega^\omega} \bigwedge_{n \in \omega} \bigvee_{k < f(n)} a_{nk}$$

In addition, measure algebras are complete Boolean algebras satisfying the countable chain condition. Von Neumann conjectured that these properties characterize measure algebras up to Boolean isomorphism. It is open whether this statement is consistent with ZFC. This is a famous problem. There has been recent progress by Dobrinen, Balcar/Jech/Pazak, and Farah/Zapletal.

A recursion-theoretic analog:

Let $\text{REC} = \{f \in \omega^\omega \mid f \text{ is recursive}\}$,
the “recursion-theoretic ground model.”

For $X \in 2^\omega$, let $\text{REC}[X] = \{g \in \omega^\omega \mid g \leq_T X\}$.

The analogous recursion-theoretic statement
would be:

For almost all $X \in 2^\omega$ with respect to the fair
coin measure on 2^ω , every $g \in \text{REC}[X]$ is
dominated by some $f \in \text{REC}$.

Unfortunately, *the above statement is false!*
(Martin, 1967, unpublished)

The proof of Martin’s result is in our paper.
(Dobrinen/Simpson, 2004)

Key definition:

(inspired by Martin's result)

Let d be a Turing degree. We say that d is *almost everywhere dominating* if for almost all $X \in 2^\omega$ and all $g \leq_T X$, there exists f of degree d such that f dominates g .

Theorems:

0 is not a.e. dominating. (Martin 1967)

$0'$ is a.e. dominating. (Kurtz 1981)

Conjecture:

d is a.e. dominating $\iff d \geq 0'$.

A variant concept:

Let d be a Turing degree. We say that d is *almost everywhere uniformly dominating* if for almost all $X \in 2^\omega$ there exists f of degree d such that f dominates all $g \leq_T X$.

By the Zero-One Law of probability theory, this implies that d is *uniformly almost everywhere dominating*, i.e., there is a fixed f of degree d which dominates all $g \leq_T X$, for almost all X .

Conjecture:

$d \geq 0' \iff d$ is uniformly a.e. dominating.

In this direction, we have the following result.

Theorem (Dobrinen/Simpson):

$d \geq 0' \iff d$ is *uniformly strongly a.e. dominating*, i.e., there is a fixed f of degree d which dominates all partial functions ψ which are partial recursive in X , for almost all X .

An alternative conjecture:

$d' \geq 0'' \iff d$ is uniformly a.e. dominating
 $\iff d$ is a.e. dominating.

In this direction, there is the following result.

Theorem (Martin 1966, ZML):

$d' \geq 0'' \iff d$ is *uniformly dominating*,
i.e., there is a fixed f of degree d which
dominates all total recursive functions.

Connection to reverse mathematics:

A well known measure-theoretic fact is that the fair coin measure μ is *regular*, i.e., for every measurable set $P \subseteq 2^\omega$ there exists an F_σ set $S \subseteq P$ such that $\mu(S) = \mu(P)$.

Here are some conjectures in the reverse mathematics of measure theory.

Conjectures:

1. $\text{ACA}_0 \iff$ for every G_δ set $P \subseteq 2^\omega$ there exists an F_σ set $S \subseteq P$ such that $\mu(S) = \mu(P)$.
2. $\text{ACA}_0 \iff$ for every G_δ set $P \subseteq 2^\omega$ and every $\epsilon > 0$, there exists a closed set $C \subseteq P$ such that $\mu(C) \geq \mu(P) - \epsilon$.

The above reverse mathematics conjectures are closely related to our a.e. domination conjectures, via the following theorems.

Theorems (Dobrinen/Simpson):

1. \mathbf{d} is uniformly a.e. dominating \iff
for all Π_2^0 sets $P \subseteq 2^\omega$,
there exists a $\Sigma_2^{0,\mathbf{d}}$ set $S \subseteq P$
such that $\mu(S) = \mu(P)$.

2. \mathbf{d} is a.e. dominating \iff
for all Π_2^0 sets $P \subseteq 2^\omega$ and for all $\epsilon > 0$,
there exists a $\Pi_1^{0,\mathbf{d}}$ set $C \subseteq P$
such that $\mu(C) \geq \mu(P) - \epsilon$.

Final remark:

If somebody proves our conjectures on a.e. domination and uniform a.e. domination, then this will probably lead to a proof of our conjectures in the reverse mathematics of measure theory.

Some of my papers are available at
<http://www.math.psu.edu/simpson/papers/>.

Transparencies for my talks are available at
<http://www.math.psu.edu/simpson/talks/>.

THE END