Muchnik and Medvedev Degrees of Π^0_1 Subsets of 2^ω

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Outline of talk:

1. The Gödel Hierarchy.

2. Reverse Mathematics and WKL_0.

3. Forcing with Π^0_1 subsets of 2^ω.

4. A symmetric ω-model of WKL_0.

5. A symmetric β-model.

6. Muchnik and Medvedev degrees of Π^0_1 sets.

7. Structural (lattice-theoretic) results.

8. Some specific Muchnik and Medvedev degrees (an invidious comparison).

9. Some classes of Muchnik degrees.

10. References.

The Gödel Hierarchy:

strong:
- ⋮
- supercompact cardinal
- ⋮
- measurable cardinal
- ⋮
- ZFC (ZF set theory with choice)
- Zermelo set theory
- simple type theory

medium:
- Z_2 (2nd order arithmetic)
- ⋮
- Π^1_2 comprehension
- Π^1_1 comprehension
- ATR_0 (arith. transfinite recursion)
- ACA_0 (arithmetical comprehension)

weak:
- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- EFA (elementary arithmetic)
- bounded arithmetic
- ⋮

Reverse Mathematics:

Let τ be a mathematical theorem. Let S_τ be the weakest natural subsystem of second order arithmetic in which τ is provable.

1. Very often, the principal axiom of S_τ is logically equivalent to τ.

2. Furthermore, only a few subsystems of second order arithmetic arise in this way.

This classification program provides
an interesting picture of the logical structure of contemporary mathematics.

It is a contribution to
foundations of mathematics (f.o.m.).

Books on Reverse Mathematics:

1.

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Perspectives in Mathematical Logic
Springer-Verlag, 1999
XIV + 445 pages

`http://www.math.psu.edu/simpson/sosoa/`

2.

S. G. Simpson (editor)
Reverse Mathematics 2001

A volume of papers by various authors,
to appear in 2001,
approximately 400 pages.

`http://www.math.psu.edu/simpson/revmath/`

An important system:

One of the most important systems
for Reverse Mathematics is WKL_0.

WKL_0 is a subsystem of second order
arithmetic.

WKL_0 includes Δ^0_1 comprehension
(i.e., closure under Turing reducibility)
and Weak König's Lemma:
(i.e., every infinite subtree of the full binary
tree has an infinite path).

Remarks on ω-models of WKL_0:

1. The ω-model

$$\mathsf{REC} = \{X \subseteq \omega : X \text{ is recursive}\}$$

is not an ω-model of WKL_0. (Kleene)

2. However, REC is the intersection of all
ω-models of WKL_0. (Kreisel, "hard core")

Remarks on ω-models of WKL_0 (continued):

3. The ω-models of WKL_0 are just the *Scott systems*, i.e., $M \subseteq P(\omega)$ such that

(a) $M \neq \emptyset$.

(b) $X, Y \in M$ implies $X \oplus Y \in M$.

(c) $X \in M$, $Y \leq_T X$ imply $Y \in M$.

(d) If $T \in M$ is an infinite subtree of $2^{<\omega}$, then there exists $X \in M$ such that X is a path through T.

Dana Scott, Algebras of sets binumerable in complete extensions of arithmetic, *Recursive Function Theory*, AMS, 1962, pages 117–121.

Remarks on ω-models of WKL_0 (continued):

4. There is a close relationship between

(a) ω-models of WKL_0, and

(b) Π^0_1 subsets of 2^ω.

The recursion-theoretic literature is extensive, with numerous articles by Jockusch, Kučera, and others. A recent survey is:

Douglas Cenzer and Jeffrey B. Remmel, Π^0_1 classes in mathematics, *Handbook of Recursive Mathematics*, North-Holland, 1998, pages 623–821.

An interesting ω-model of WKL_0:

Let $\mathcal{P}$ be the nonempty Π^0_1 subsets of 2^ω, ordered by inclusion. Forcing with $\mathcal{P}$ is known as Jockusch/Soare forcing.

Lemma (Simpson 2000). Let X be J/S generic. Suppose $Y \leq_T X$. Then (i) Y is J/S generic, and (ii) X is J/S generic relative to Y.

Theorem (Simpson 2000). There is an ω-model M of WKL_0 with the following property: For all $X, Y \in M$, X is definable from Y in M if and only if X is Turing reducible to Y.

Proof. M is obtained by iterated J/S forcing. We have

$$M = \mathsf{REC}[X_1, X_2, \ldots, X_n, \ldots]$$

where, for all n, X_{n+1} is J/S generic over $\mathsf{REC}[X_1, \ldots, X_n]$. To show that M has the desired property, we use symmetry arguments based on the Recursion Theorem.

Foundational significance of M:

The above ω-model, M, represents a compromise between the conflicting needs of

(a) recursive mathematics (“everything is computable”)

and

(b) classical rigorous mathematics as developed in WKL_0 (“every continuous real-valued function on $[0,1]$ attains a maximum”, “every countable commutative ring has a prime ideal”, etc etc).

Namely, M contains enough nonrecursive objects for WKL_0 to hold, yet the recursive objects form the “definable core” of M.

Foundational significance (continued):

More generally, consider the scheme

> (*) For all X and Y, if X is definable from Y then X is recursive in Y

in the language of second order arithmetic.

Often in mathematics, under some assumptions on a given countably coded object X, there exists a unique countably coded object Y having some property stated in terms of X. In this situation, (*) implies that Y is Turing computable from X. This is of obvious f.o.m. significance.

Simpson 2000 shows that, for every countable model of WKL_0, there exists a countable model of $\mathsf{WKL}_0 + (*)$ with the same first order part.

Thus $\mathsf{WKL}_0 + (*)$ is conservative over WKL_0 for first order arithmetical sentences.

Earlier results of Friedman:

In an unpublished 1974 manuscript, Friedman obtained (by a different method) an ω-model M of WKL_0 with the following property: For all $X \in M$, X is definable in M if and only if X is recursive.

In the same 1974 manuscript, Friedman proved another result which stands in contradiction to my theorem above, concerning relative definability and relative recursiveness. Friedman's proof of this other result is erroneous.

A Π^0_1 **set of** ω**-models of** WKL$_0$**:**

Theorem (Simpson 2000). There is a nonempty Π^0_1 subset of 2^ω, P, such that:

1. For all $X \in P$, $\{(X)_n : n \in \omega\}$ is a countable ω-model of WKL$_0$, and every countable ω-model of WKL$_0$ occurs in this way.

2. For all nonempty Π^0_1 sets $P_1, P_2 \subseteq P$ we can find a recursive homeomorphism

$$F : P_1 \cong P_2$$

such that for all $X \in P_1$ and $Y \in P_2$, if $F(X) = Y$ then

$$\{(X)_n : n \in \omega\} = \{(Y)_n : n \in \omega\} \ .$$

The proof uses an idea of Pour-El/Kripke 1967.

Hyperarithmetical analogs:

Theorem (Simpson 2000). There is a countable β-model M such that, for all $X, Y \in M$, X is definable from Y in M if and only if X is hyperarithmetical in Y.

In the language of second order arithmetic, consider the scheme

> (**) for all X, Y, if X is definable from Y, then X is hyperarithmetical in Y.

Theorem (Simpson 2000).

1. $\mathsf{ATR}_0 + (**)$ is conservative over ATR_0 for Σ^1_2 sentences.
2. $\Pi^1_\infty\text{-}\mathsf{TI}_0 + (**)$ is conservative over $\Pi^1_\infty\text{-}\mathsf{TI}_0$ for Σ^1_2 sentences.

Two new structures in recursion theory:

Recall that $\mathcal{P}$ is the set of nonempty Π^0_1 subsets of 2^ω.

$\mathcal{P}_w$ ($\mathcal{P}_M$) consists of the Muchnik (Medvedev) degrees of members of $\mathcal{P}$, ordered by Muchnik (Medvedev) reducibility.

P is Muchnik reducible to Q ($P \leq_w Q$) if for all $Y \in Q$ there exists $X \in P$ such that $X \leq_T Y$.

P is Medvedev reducible to Q ($P \leq_M Q$) if there exists a recursive functional $F : Q \to P$.

Note: $\leq_M$ is a uniform version of $\leq_w$.

$\mathcal{P}_w$ and $\mathcal{P}_M$ are countable distributive lattices with 0 and 1.

The lattice operations are given by

$$P \times Q = \{X \oplus Y : X \in P, Y \in Q\}$$

(least upper bound)

$$P + Q = \{\langle 0 \rangle^\frown X : X \in P\} \cup \{\langle 1 \rangle^\frown Y : Y \in Q\}$$

(greatest lower bound).

$P \equiv 0$ in $\mathcal{P}_w$ if and only if $P \cap \mathsf{REC} \neq \emptyset$.

$P \equiv 0$ in $\mathcal{P}_M$ if and only if $P \cap \mathsf{REC} \neq \emptyset$.

$P \equiv 1$ in $\mathcal{P}_w$, i.e., P is *Muchnik complete*, if and only if the Turing degrees of members of P are exactly the Turing degrees of complete extensions of PA. (Simpson 2001)

$P \equiv 1$ in $\mathcal{P}_M$, i.e., P is *Medvedev complete*, if and only if P is recursively homeomorphic to the set of complete extensions of PA. (Simpson 2000)

Connection with Lindenbaum algebras:

Stone duality gives a 1-1 correspondence

$$P \longleftrightarrow B_P$$

between *members of* $\mathcal{P}$ (i.e., nonempty Π^0_1 subsets of 2^ω) and *Lindenbaum sentence algebras of r.e. theories* (i.e., Boolean algebras of the form B/I, where B is the countable free Boolean algebra, and I is an r.e. ideal in B).

Moreover, this correspondence is *functorial*.

Namely, recursive functionals $F : Q \to P$ (i.e., Medvedev reductions) correspond to recursive homomorphisms $B_F : B_P \to B_Q$.

This provides an alternative way to view Medvedev reducibility: $P \leq_M Q$ if and only if there exists a recursive homomorphism $f : B_P \to B_Q$.

Structural (lattice-theoretic) results:

Trivially $P, Q > 0$ implies $P + Q > 0$, but we do not know whether $P, Q < 1$ implies $P \times Q < 1$.

In $\mathcal{P}_w$, for every $P > 0$, every countable distributive lattice is lattice embeddable below P. For $\mathcal{P}_M$ we have partial results in this direction.

To construct our lattice embeddings, we use infinitary "almost lattice" operations, defined in such a way that, if $\langle P_i : i \in \omega \rangle$ is a recursive sequence of members of $\mathcal{P}$, then

$$\prod_{i=0}^{\infty} P_i \qquad \text{and} \qquad \sum_{i=0}^{\infty} P_i$$

are again members of $\mathcal{P}$. We also use a finite injury priority argument a la Martin/Pour-El 1970 and Jockusch/Soare 1972. To push the embeddings below P, we use a Sacks preservation strategy.

This is ongoing joint work with my Ph. D. student Stephen Binns.

Structural results (continued):

Corollary. In $\mathcal{P}_w$, for all $P >_w 0$ there exists Q such that $P >_w Q >_w 0$.
(nonexistence of minimal Muchnik degrees)

Corollary. In $\mathcal{P}_M$, for all $P >_M 0$ there exists Q such that $P >_M Q >_M 0$.
(nonexistence of minimal Medvedev degrees)

The last corollary was also obtained by Douglas Cenzer and Peter Hinman, using a different method: index sets.

Problem area:

Study structural properties of the countable distributive lattices $\mathcal{P}_w$ and $\mathcal{P}_M$. Lattice embeddings, extensions of embeddings, quotient lattices, cupping and capping, automorphisms, definability, decidability, etc.

An invidious comparison:

In some ways, the study of $\mathcal{P}_w$ and $\mathcal{P}_M$ parallels the study of $\mathcal{R}_T$, the Turing degrees of recursively enumerable subsets of ω.

Analogy: $$\frac{\mathcal{P}_w}{\mathcal{R}_T} = \frac{\mathsf{WKL}_0}{\mathsf{ACA}_0}$$

A regrettable aspect of $\mathcal{R}_T$ is that there are **no specific known examples** of recursively enumerable Turing degrees $\neq 0, 0'$. (See the extensive FOM discussion of July 1999, in the aftermath of the Boulder meeting.)

In this respect, $\mathcal{P}_w$ and $\mathcal{P}_M$ are **much better**.

Invidious comparison (continued):

For example, we have:

Theorem. The set of Muchnik degrees of Π^0_1 subsets of 2^ω of positive measure contains a maximum degree. This particular Muchnik degree is $\neq 0, 1$.

Question. What about Medvedev degrees?

The theorem follows from three known results.

1. $\{X : X$ is 1-random$\}$ is Σ^0_2 and of measure one. (Martin-Löf 1966)

2. $\{X : \exists Y \leq_T X\,(Y$ separates a recursively inseparable pair of r.e. sets$)\}$ is of measure zero. (Jockusch/Soare 1972)

3. If $P \in \mathcal{P}$ is of positive measure, then for all 1-random X there exists k such that $X^{(k)} = \lambda n.X(n+k) \in P$. (Kučera 1985)

A related but apparently new result:

Theorem (Simpson 2000). If X is 1-random and hyperimmune-free, then no $Y \leq_T X$ separates a recursively inseparable pair of r.e. sets.

Other related results:

1. If X is 1-random and of r.e. Turing degree, then X is Turing complete. (Kučera 1985)

2. $\{X : X$ is hyperimmune-free$\}$ is of measure zero. (Martin 1967, unpublished)

Foundational significance:

All of these results are informative with respect to ω-models of WWKL_0. WWKL_0 is a subsystem of second order arithmetic which arises in the Reverse Mathematics of measure theory. (Yu/Simpson 1990)

Some specific Medvedev degrees $\neq 0, 1$:

For $k \geq 2$ let DNR_k be the set of k-valued DNR functions. Each DNR_k is recursively homeomorphic to a member of $\mathcal{P}$. DNR_2 is Medvedev complete. In $\mathcal{P}_M$ we have

$$\mathsf{DNR}_2 >_M \mathsf{DNR}_3 >_M \cdots >_M \sum_{k=2}^{\infty} \mathsf{DNR}_k \,.$$

All of these Medvedev degrees are Muchnik complete. (Jockusch 1989)

Problem area:

Find additional natural examples of Medvedev and Muchnik degrees $\neq 0, 1$.

Experience suggests that natural examples could be of significance for f.o.m.

A related problem of Reverse Mathematics:

Let DNR(k) be the statement that for all X there exists a k-valued DNR function relative to X. It is known that, for each $k \geq 2$, DNR(k) is equivalent to Weak König's Lemma over RCA_0. Is $\exists k\,(k \geq 2 \wedge \mathrm{DNR}(k))$ equivalent to Weak König's Lemma over RCA_0?

This has a bearing on graph coloring problems in Reverse Mathematics. See two recent papers of James H. Schmerl, to appear in *MLQ* and *Reverse Mathematics 2001*.

Another problem area:

One may study properties of interesting subsets of $\mathcal{P}_w$ and $\mathcal{P}_M$. For example, we may consider Muchnik and Medvedev degrees of $P \in \mathcal{P}$ with the following special properties:

1. P is of positive measure.

2. P is *thin*, i.e., for all Π^0_1 sets $Q \subseteq P$ there exists a clopen set $U \subseteq 2^\omega$ such that $P \cap U = Q$. (See also the recent paper of Cholak/Coles/Downey/Herrmann.)

3. P is *separating*, i.e.,

$$P = \{X \in 2^\omega : X \text{ separates } A, B\}$$

where A, B is a disjoint pair of r.e. sets.

These classes of Medvedev and Muchnik degrees are related in interesting ways.

Theorem (Simpson 2001). Let $P \subseteq 2^\omega$ be Π^0_1 of positive measure of maximum Muchnik degree. Let $Q \not\equiv_w 0$ be a thin Π^0_1 set. Then P and Q are Muchnik incomparable, i.e., $P \not\leq_w Q$ and $Q \not\leq_w P$.

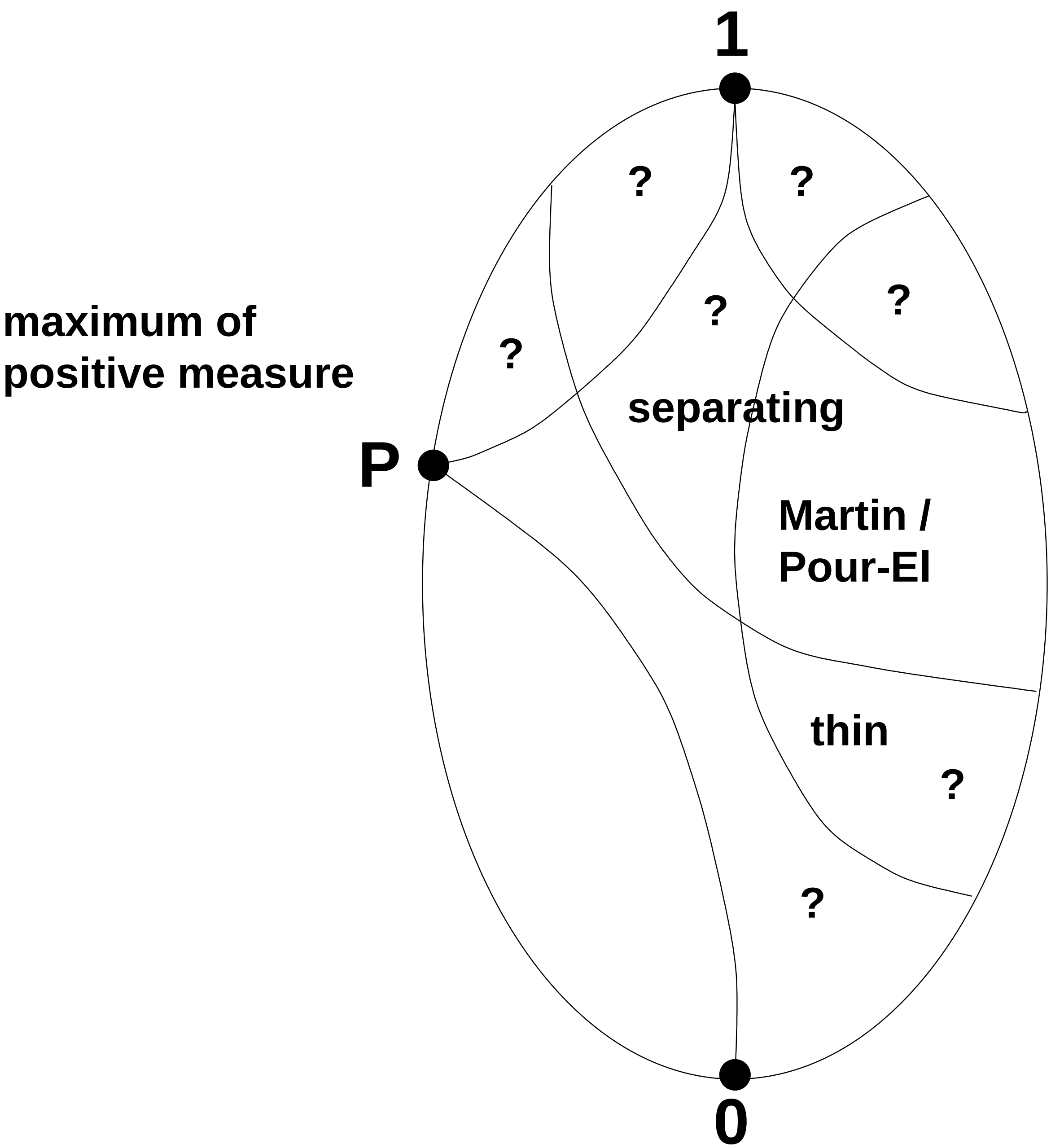
Also, if P is as above and $Q \not\equiv_w 0$ is separating, then $Q \not\leq_w P$. (Jockusch/Soare 1972)

Theorem (Simpson 2001). Let P be as above. Then P is non-capping in $\mathcal{P}_w$. I.e., there do not exist $P_1, P_2 >_w P$ such that $P \equiv_w P_1 + P_2$, the infimum of P_1 and P_2.

A lemma used in proving these theorems:

Lemma. If $P, Q \in \mathcal{P}$ and $P \leq_w Q$, then there exists $R \subseteq Q$, $R \in \mathcal{P}$, such that $P \leq_M R$.

A picture of the Muchnik lattice $\mathcal{P}_w$:



References:

Stephen G. Simpson, Kazuyuki Tanaka, and Takeshi Yamazaki, Some conservation results on weak König's lemma, preprint, February 2000, 26 pages, to appear in APAL.

Stephen G. Simpson, Π^0_1 sets and models of WKL_0, preprint, April 2000, 28 pages, to appear in *Reverse Mathematics 2001*.

Stephen G. Simpson, A symmetric β-model, preprint, May 2000, 7 pages, to appear.

Stephen Binns and Stephen G. Simpson, Medvedev and Muchnik degrees of Π^0_1 subsets of 2^ω, in preparation.

Stephen G. Simpson, Some examples of Muchnik degrees of Π^0_1 subsets of 2^ω, in preparation.

Some of my papers are available at
`http://www.math.psu.edu/simpson/papers/`.

Transparencies for my talks are available at
`http://www.math.psu.edu/simpson/talks/`.

THE END