Degrees of Unsolvability

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Turing 1936: unsolvability of the Halting Problem.

But, how much unsolvability is inherent in the Halting Problem?

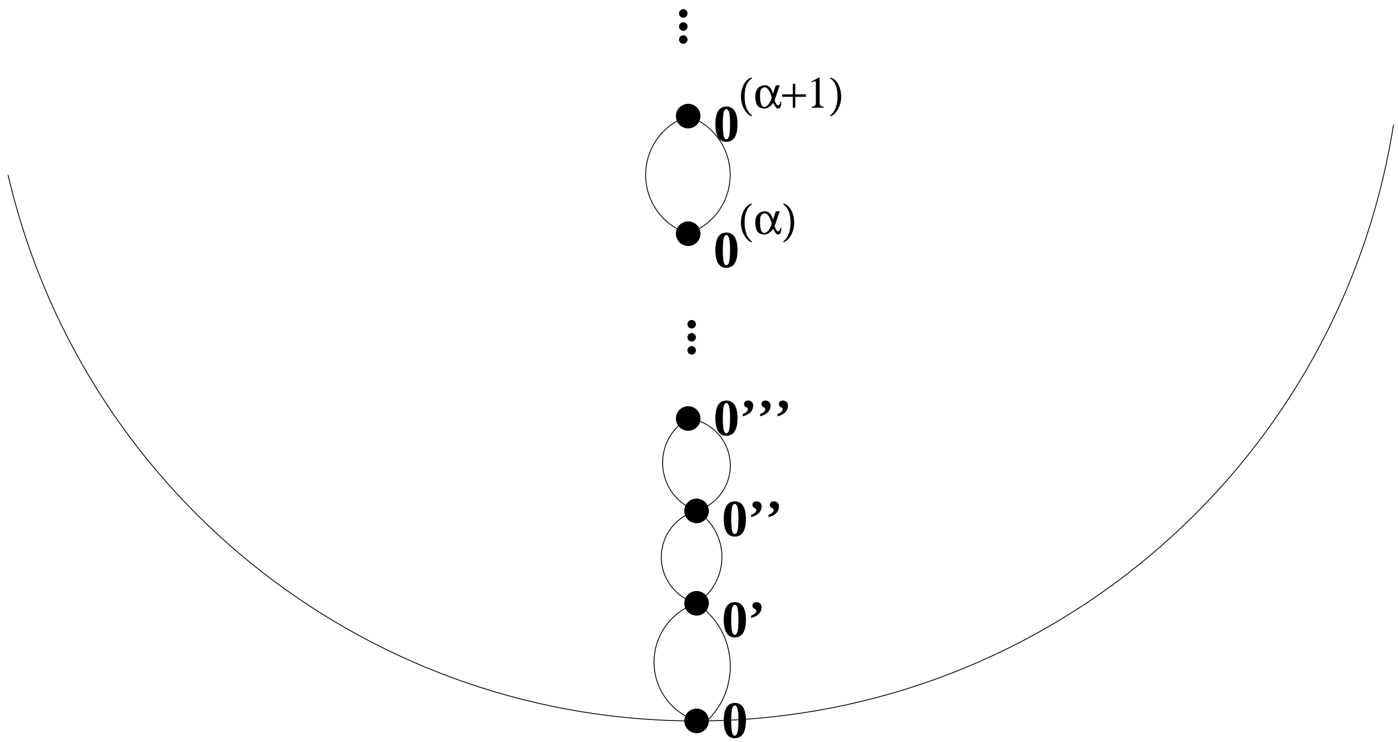
We consider various unsolvable problems, and we measure and compare the amount or degree of unsolvability which is inherent in them.

The Halting Problem has the same degree as the decision problem for Peano Arithmetic: to decide whether a given sentence is provable in PA or not. This degree is denoted $\mathbf{0}'$.

Post 1948, Kleene/Post 1954: decision problems.
A Turing degree is the degree of a decision problem.
$\mathcal{D}_\mathrm{T}$ = the semilattice of all Turing degrees.
$\mathcal{E}_\mathrm{T}$ = the sub-semilattice consisting of the recursively enumerable Turing degrees, i.e., the degrees of decision problems for recursively axiomatizable theories.

The top and bottom degrees in $\mathcal{E}_\mathrm{T}$ are $\mathbf{0}'$ and $\mathbf{0}$.
Beyond $\mathcal{E}_\mathrm{T}$, there is a natural hierarchy of Turing degrees $\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \mathbf{0}''' < \cdots < \mathbf{0}^{(n)} < \cdots < \mathbf{0}^{(\alpha)} < \mathbf{0}^{(\alpha+1)} < \cdots$ where n is a natural number, α is a transfinite ordinal number, and $\mathbf{d} \mapsto \mathbf{d}' : \mathcal{D}_\mathrm{T} \to \mathcal{D}_\mathrm{T}$ is the Turing jump operator. If $\mathbf{d}$ is the degree of a decision problem D, then $\mathbf{d}'$ is the degree of the Halting Problem relative to the oracle D.

A picture of $\mathcal{D}_\mathrm{T}$, the semilattice of all Turing degrees.



In this picture, the black dots denote Turing degrees
$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \mathbf{0}''' < \cdots < \mathbf{0}^{(\alpha)} < \mathbf{0}^{(\alpha+1)} < \cdots$,
i.e., finite and transfinite iterates of the Turing jump operator.

The <u>only known specific, natural examples of Turing degrees</u> are of the form $\mathbf{0}^{(\alpha)}$ where α is a specific, natural, countable ordinal number. E.g., $\alpha = 1$, or $\alpha = 3$, or $\alpha = \omega$, or $\alpha = \omega^\omega$, or $\alpha = \varepsilon_0$, or $\alpha = \omega_1^{\mathrm{CK}}$.

I began studying degrees of unsolvability in the late 1960's under my future thesis adviser, Gerald Sacks. In his book *Degrees of Unsolvability* I found certain infinite families of degrees in $\mathcal{E}_{\mathrm{T}}$, but there were no specific, natural examples of degrees in $\mathcal{E}_{\mathrm{T}}$, other than $\mathbf{0}$ and $\mathbf{0}'$.

I asked Gerald about this. His answer was:

"All examples are misleading."

Later, on page 4 of his book *Saturated Model Theory*, he elaborated:

> It is no accident that the book suffers from a shortage of examples. As a rule examples are presented by authors in the hope of clarifying universal concepts, but all examples of the universal, since they must of necessity be particular and so partake of the individual, are misleading.

These comments made me think hard. I had great respect for recursively enumerable degrees and for Gerald's insights and opinions concerning them. However, I had already studied a fair amount of graduate level mathematics, so I knew very well that most branches of mathematics are motivated and nurtured by a rich stock of examples. Therefore, as regards $\mathcal{E}_{\mathrm{T}}$, it seemed important to somehow remedy the lack of examples of r.e. Turing degrees.

I started by trying to find at least one specific, natural example of an r.e. Turing degree strictly between $\mathbf{0}$ and $\mathbf{0}'$. However, I learned early on that Turing degree determinacy presents a serious obstacle. Therefore, I instead thought about how to "tweak" $\mathcal{E}_{\mathrm{T}}$ to get a better structure. Inspired by the reverse mathematics of measure theory, I finally found the right structure, which I call $\mathcal{E}_{\mathrm{w}}$.

$\mathcal{E}_{\mathrm{w}}$ is the lattice of Muchnik degrees of nonempty Π^0_1 subclasses of $\{0,1\}^{\mathbb{N}}$. It is a sublattice of the lattice $\mathcal{D}_{\mathrm{w}}$ of all Muchnik degrees. Just as Turing degrees are the degrees of decision problems, so Muchnik degrees are the degrees of mass problems.

In this talk I shall explain mass problems and Muchnik degrees, and I shall give an indication of what has been discovered about $\mathcal{D}_{\mathrm{w}}$ and $\mathcal{E}_{\mathrm{w}}$. The main points that I want to make are:

1. $\mathcal{E}_{\mathrm{w}}$ is closely analogous to $\mathcal{E}_{\mathrm{T}}$.

2. $\mathcal{E}_{\mathrm{w}}$ includes a copy of $\mathcal{E}_{\mathrm{T}}$.

3. $\mathcal{E}_{\mathrm{w}}$ is in some ways better-behaved than $\mathcal{E}_{\mathrm{T}}$.

4. $\mathcal{E}_{\mathrm{w}}$, unlike $\mathcal{E}_{\mathrm{T}}$, contains a tremendous wealth of specific, natural degrees which are closely related to interesting subjects such as reverse mathematics and Kolmogorov complexity.

Five books with the same title.

1. Gerald E. Sacks, Degrees of Unsolvability, Princeton University Press, 1963, 55, Annals of Mathematics Studies, IX + 174 pages.

2. Joseph R. Shoenfield, Degrees of Unsolvability, North-Holland, 1971, 2, North-Holland Mathematics Studies, VIII + 111 pages.

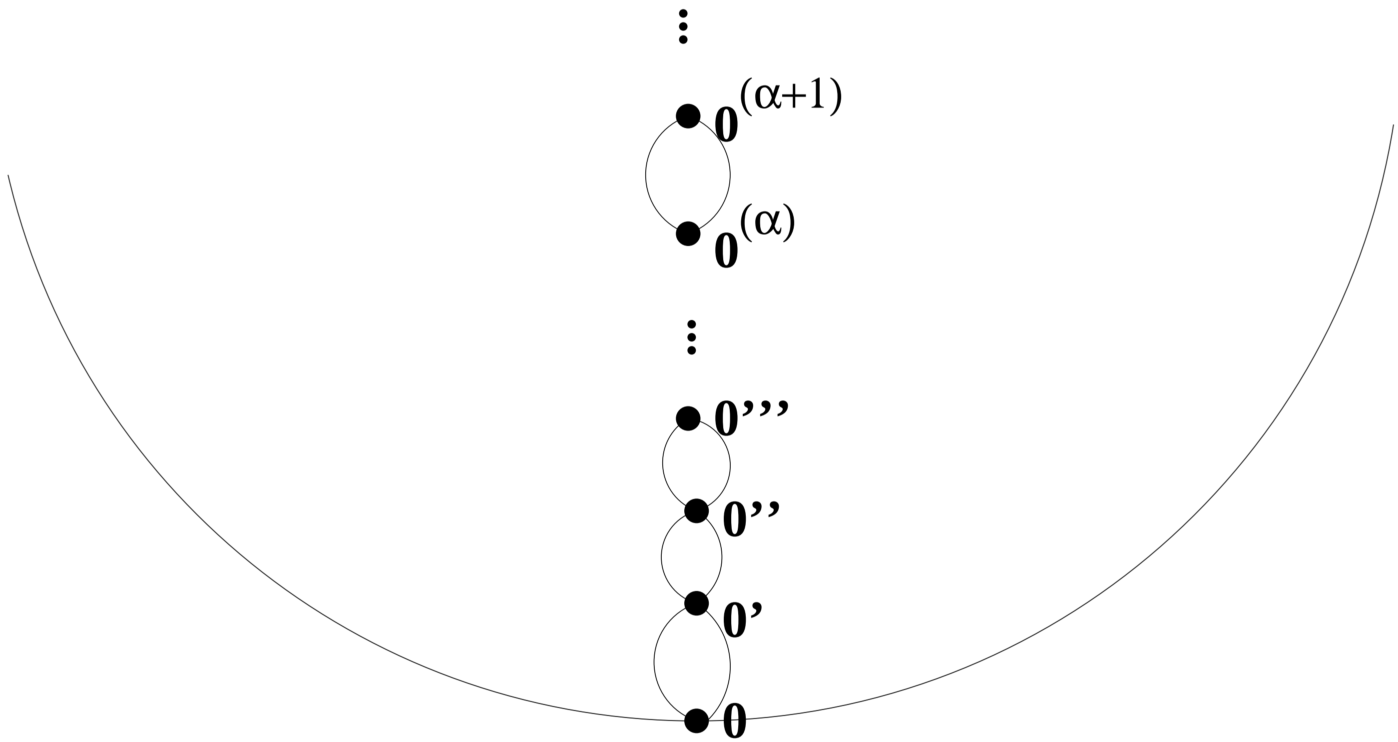
3. Richard L. Epstein, Degrees of Unsolvability: Structure and Theory, Lecture Notes in Mathematics, 759, Springer-Verlag, 1979, XIV + 240 pages.

4. Manuel Lerman, Degrees of Unsolvability, Springer-Verlag, 1983, Perspectives in Mathematical Logic, XIII + 307 pages.

5. Stephen G. Simpson, Degrees of Unsolvability, in preparation.

However, my book will be different from the others.
Namely, it will emphasize <u>specific natural examples</u> of degrees, including but not limited to iterates of the jump operator
$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \mathbf{0}''' \cdots < \mathbf{0}^{(\alpha)} < \mathbf{0}^{(\alpha+1)} < \cdots$.

A picture of $\mathcal{D}_\mathrm{T}$, the semilattice of all Turing degrees.



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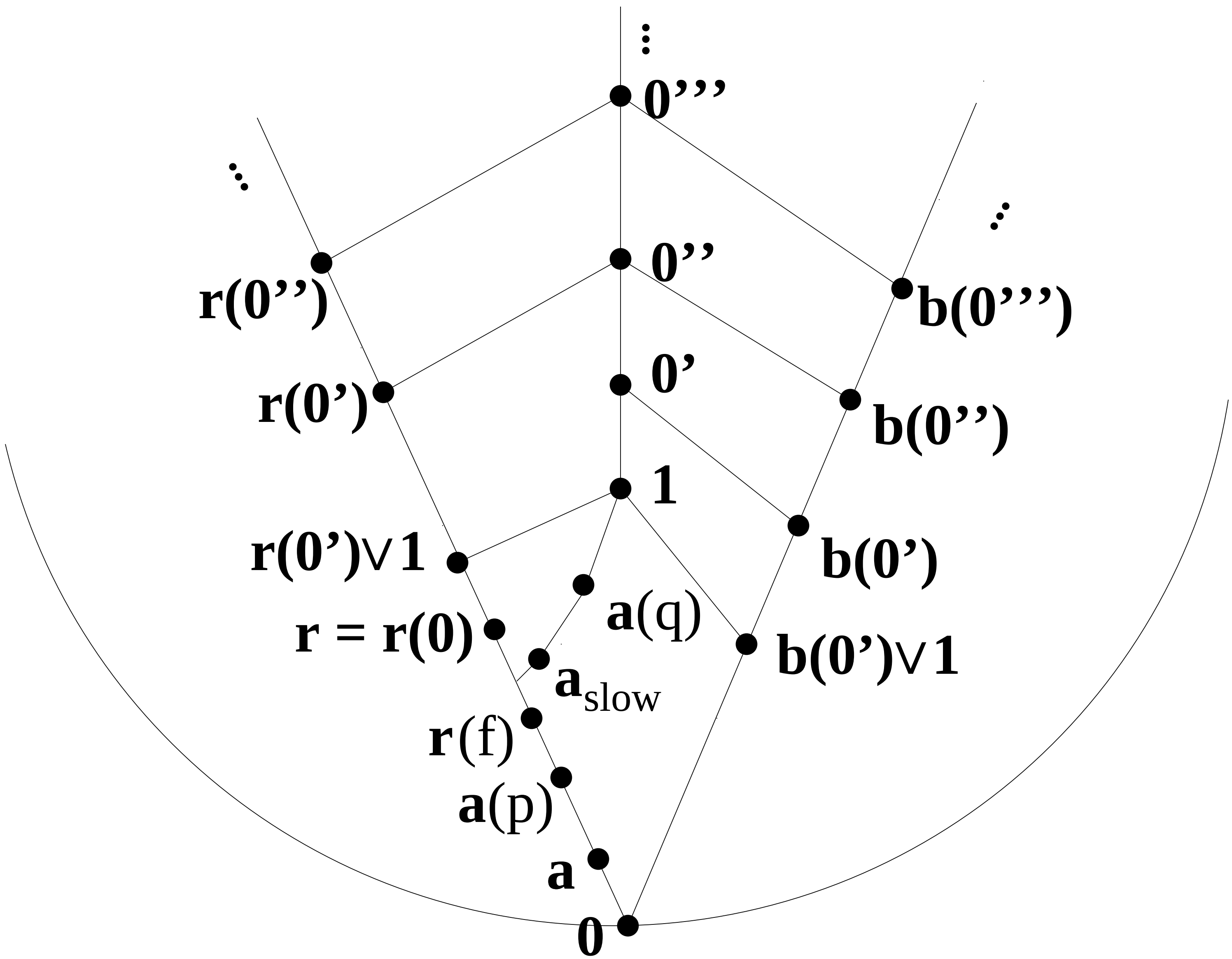
i.e., finite and transfinite iterates of the Turing jump operator.

The only known specific, natural examples of Turing degrees are of the form $\mathbf{0}^{(\alpha)}$ where α is a specific, natural, countable ordinal number. E.g., $\alpha = 1$, or $\alpha = 3$, or $\alpha = \omega$, or $\alpha = \omega^\omega$, or $\alpha = \varepsilon_0$, or $\alpha = \omega_1^{\mathrm{CK}}$.

A picture of $\mathcal{D}_{\mathrm{w}}$, the lattice of all Muchnik degrees.



A key insight: There are many interesting unsolvable problems which are not decision problems and do not have Turing degrees.
Example: the completion problem for Peano Arithmetic, denoted CPA. This unsolvable problem is not equivalent to any decision problem. However, it is still possible to assign a degree to CPA. So this degree, denoted $\mathbf{1}$, cannot be a Turing degree. In other words, $\mathbf{1} \notin \mathcal{D}_{\mathsf{T}}$.

Muchnik 1963, Simpson 1999: mass problems.
A Muchnik degree is the degree of a mass problem.
$\mathcal{D}_{\mathsf{w}}$ = the lattice of all Muchnik degrees.
$\mathcal{E}_{\mathsf{w}}$ = the lattice of Muchnik degrees of nonempty Π^0_1 classes $P \subseteq \{0,1\}^{\mathbb{N}}$.

An example of a mass problem is the completion problem CPA. The Muchnik degree of CPA is $\mathbf{1}$. For a long time Muchnik degrees were unknown in the USA, perhaps because of the Cold War.

The lattice $\mathcal{D}_{\mathsf{w}}$ includes the semilattice $\mathcal{D}_{\mathsf{T}}$, and there is an analogy

$$\frac{\mathcal{E}_{\mathsf{T}}}{\mathcal{D}_{\mathsf{T}}} = \frac{\mathcal{E}_{\mathsf{w}}}{\mathcal{D}_{\mathsf{w}}}.$$

Just as $\mathcal{E}_{\mathsf{T}}$ is a sub-semilattice of $\mathcal{D}_{\mathsf{T}}$, so $\mathcal{E}_{\mathsf{w}}$ is a sublattice of $\mathcal{D}_{\mathsf{w}}$.
The top and bottom degrees in $\mathcal{E}_{\mathsf{T}}$ are $\mathbf{0}'$ and $\mathbf{0}$.
The top and bottom degrees in $\mathcal{E}_{\mathsf{w}}$ are $\mathbf{1}$ and $\mathbf{0}$.
And $\mathbf{0} < \mathbf{1} < \mathbf{0}'$, but the only Turing degree in $\mathcal{E}_{\mathsf{w}}$ is $\mathbf{0}$.

What are mass problems and Muchnik degrees?

For purposes of this talk, a mass problem is any set $P \subseteq \mathbb{N}^{\mathbb{N}}$, viewed as the "problem" of "finding" or "computing" at least one member of P. Thus, a mass problem $P \subseteq \mathbb{N}^{\mathbb{N}}$ is solvable if at least one member of P is recursive, i.e., $P \cap \mathsf{Rec} \neq \emptyset$. And for mass problems $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$, we say that P is reducible to Q, written $P \leq_{\mathsf{w}} Q$, if $(\forall Y \in Q)\,(\exists X \in P)\,(X \leq_{\mathsf{T}} Y)$. I.e., every "solution" of Q can be used as an oracle to "compute" at least one "solution" of P. The Muchnik degree of P, denoted $\deg_{\mathsf{w}}(P)$, is the equivalence class of P under mutual reducibility.

The lattice $\mathcal{D}_{\mathsf{w}}$ of all Muchnik degrees may be viewed as the completion of the semilattice $\mathcal{D}_{\mathsf{T}}$ of all Turing degrees, in much the same way as the real numbers are the completion of the rational numbers.

Namely, we have a natural embedding $\deg_{\mathsf{T}}(X) \mapsto \deg_{\mathsf{w}}(\{Y \mid X \leq_{\mathsf{T}} Y\}) \colon \mathcal{D}_{\mathsf{T}} \to \widehat{\mathcal{D}_{\mathsf{T}}} = \mathcal{D}_{\mathsf{w}}$. This is an instance of a very general construction going back to Birkhoff 1937 and Alexandrov 1937. For any partially ordered set $\mathcal{K}$, there is natural embedding $x \mapsto \{y \in \mathcal{K} \mid x \leq y\} \colon \mathcal{K} \to \widehat{\mathcal{K}}$, where $\widehat{\mathcal{K}}$ consists of all upwardly closed subsets of $\mathcal{K}$, ordered by reverse inclusion. Note that $\widehat{\mathcal{K}}$ is a complete and completely distributive lattice.

We know many examples of specific, natural, real numbers which are not rational numbers: $\sqrt{2}$, e, π, etc.

Similarly, there are many specific, natural Muchnik degrees which are not Turing degrees:

$$\mathbf{r} = \mathbf{r}(\mathbf{0}) < \mathbf{r}(\mathbf{0}') < \mathbf{r}(\mathbf{0}'') < \cdots < \mathbf{r}(\mathbf{0}^{(\omega)}) < \cdots < \infty,$$

$$\mathbf{r} = \mathbf{r}(\cdot) > \mathbf{r}(\cdot/2) > \mathbf{r}(\sqrt[2]{\cdot}) > \mathbf{r}(\log_2 \cdot) > \cdots > \mathbf{r}(\mathsf{Rec}) > \mathbf{a},$$

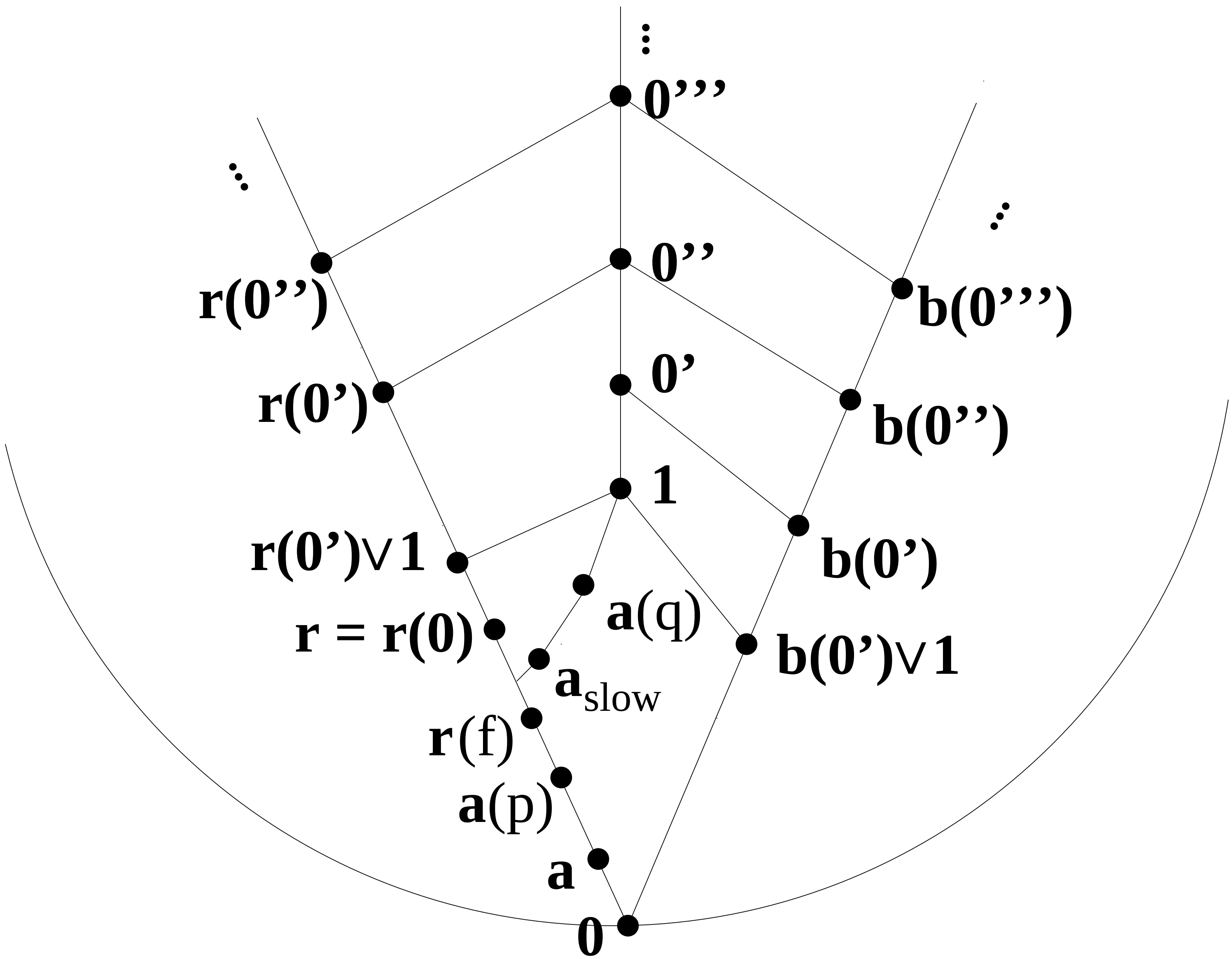
$$\mathbf{a} < \mathbf{a}(\mathsf{Rec}) < \cdots < \mathbf{a}(2^{\cdot}) < \mathbf{a}(\cdot^{2}) < \cdots,$$

$$\mathbf{a}_{\mathsf{slow}} < \cdots < \mathbf{a}(\cdot) < \mathbf{a}(\cdot/2) < \mathbf{a}(\sqrt[2]{\cdot}) < \mathbf{a}(\log_2 \cdot) < \cdots < \mathbf{1},$$

$$\mathbf{b}(\mathbf{0}') < \mathbf{b}(\mathbf{0}'') < \mathbf{b}(\mathbf{0}''') < \cdots < \mathbf{b}(\mathbf{0}^{(\omega)}) < \cdots < \infty.$$

We shall explain these examples of Muchnik degrees in the lattice $\mathcal{D}_{\mathsf{w}}$. Some of them belong to the sublattice $\mathcal{E}_{\mathsf{w}}$ and some do not.

A picture of $\mathcal{D}_W$.



$\mathbf{0} = \deg_{\mathsf{w}}(0)$, and $\mathbf{0}^{(\alpha)}$ = the αth Turing jump of $\mathbf{0}$. Thus $\mathbf{0} = \mathbf{0}^{(0)}$, $\mathbf{0}' = \mathbf{0}^{(1)}$, $\mathbf{0}'' = \mathbf{0}^{(2)}$, $\mathbf{0}''' = \mathbf{0}^{(3)}$, etc., and all of these are Turing degrees.

$\mathbf{1} = \deg_{\mathsf{w}}(\mathsf{CPA})$ where CPA is the problem of finding a completion of Peano Arithmetic. Instead of PA we could use Z_2, or ZFC, etc.

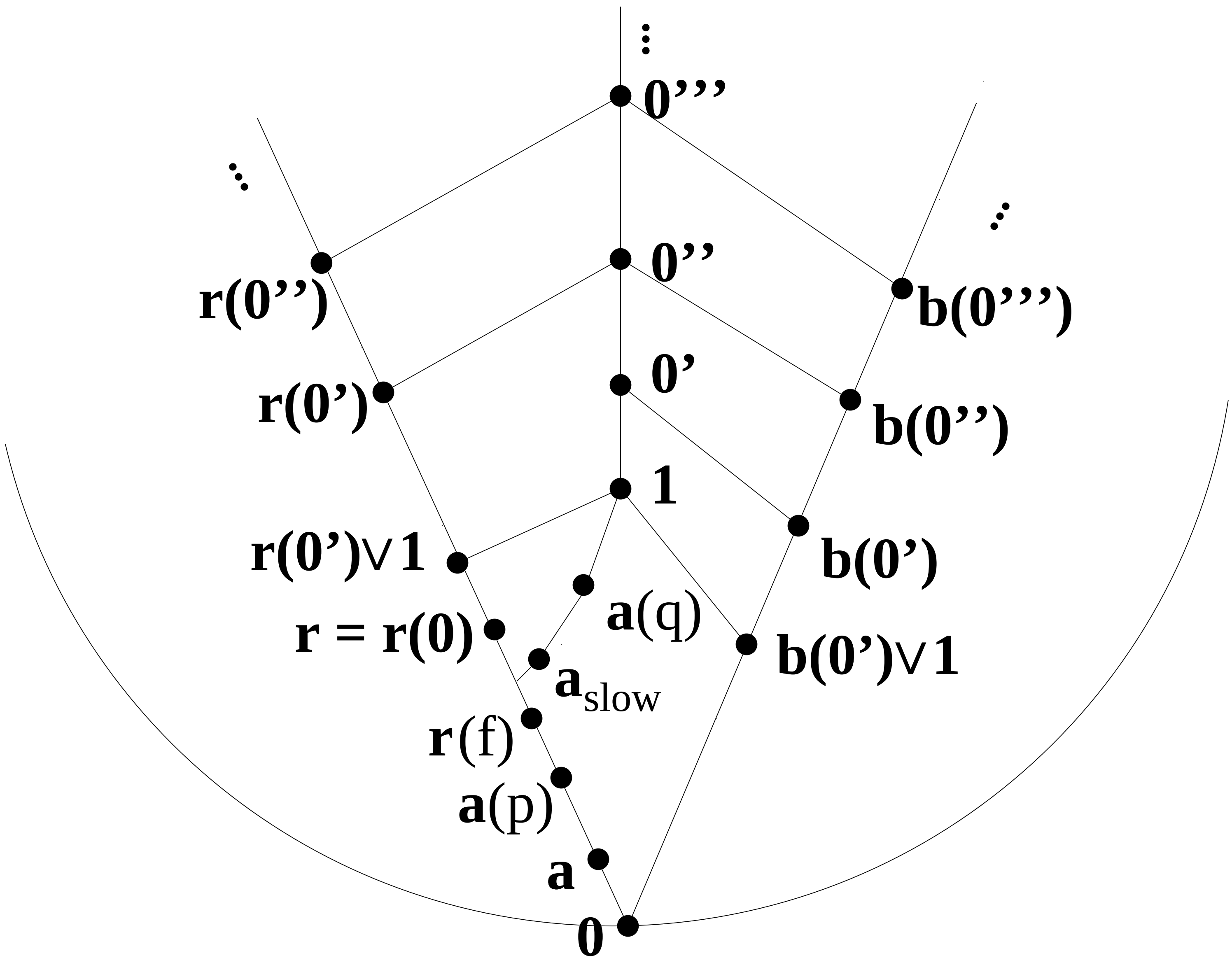
$\mathbf{r} = \deg_{\mathsf{w}}(\mathsf{MLR})$ where MLR is the problem of finding at least one infinite sequence $Z \in \{0,1\}^{\mathbb{N}}$ which is random in the sense of Martin-Löf. This degree is implicit in the reverse mathematics of measure theory.

More generally, for any Turing degree $\mathbf{x} = \deg_{\mathsf{T}}(X)$, we write $\mathbf{r}(\mathbf{x}) = \deg_{\mathsf{w}}(\mathsf{MLR}^X)$ where $\mathsf{MLR}^X = \{Z \in \{0,1\}^{\mathbb{N}} \mid Z$ is random relative to $X\}$.

Also, we write $\mathbf{b}(\mathbf{x}) = \deg_{\mathsf{w}}(\{Y \mid \mathsf{MLR}^X \leq_{\mathsf{w}} \mathsf{MLR}^Y\})$. Note that $\mathsf{MLR}^X \leq_{\mathsf{w}} \mathsf{MLR}^Y$ if and only if X is LR-reducible to Y in the sense of Nies.

The degrees $\mathbf{r}(\mathbf{0}^{(\alpha)})$ and $\mathbf{b}(\mathbf{0}^{(\alpha)})$ are implicit in the reverse mathematics of measure theory. For instance, $\mathbf{b}(\mathbf{0}^{(\alpha)})$ = the Muchnik degree of this problem: to find a Turing oracle Y such that every $\mathbf{\Sigma}^0_{\alpha+2}$ set $S \subseteq \mathbb{R}$ has a $\mathbf{\Sigma}^{0,Y}_2$ subset of the same Lebesgue measure.

A picture of $\mathcal{D}_W$.



More on the sublattice $\mathcal{E}_{\mathsf{w}}$.

Recall that $\mathcal{E}_{\mathsf{T}} = \{\deg_{\mathsf{T}}(X) \mid X \subseteq \mathbb{N}$ is a $\boldsymbol{\Sigma}^0_1$ decision problem$\}$, and the top and bottom degrees of $\mathcal{E}_{\mathsf{T}}$ are $\mathbf{0}'$ and $\mathbf{0}$.

Similarly, $\mathcal{E}_{\mathsf{w}} = \{\deg_{\mathsf{w}}(P) \mid P \subseteq \{0,1\}^{\mathbb{N}}$ is a nonempty Π^0_1 class$\}$, and the top and bottom degrees of $\mathcal{E}_{\mathsf{w}}$ are $\mathbf{1}$ and $\mathbf{0}$.

Note the analogy: $\mathcal{E}_{\mathsf{T}}$ (respectively $\mathcal{E}_{\mathsf{w}}$) is the sub-semilattice of $\mathcal{D}_{\mathsf{T}}$ (respectively, the sublattice of $\mathcal{D}_{\mathsf{w}}$) at the lowest level of the arithmetical hierarchy.

However, the only degree in $\mathcal{E}_{\mathsf{T}} \cap \mathcal{E}_{\mathsf{w}}$ is $\mathbf{0}$.

There is an obvious standard embedding of $\mathcal{D}_{\mathsf{T}}$ into $\mathcal{D}_{\mathsf{w}}$, namely $\deg_{\mathsf{T}}(X) \mapsto \deg_{\mathsf{w}}(\{X\}) = \deg_{\mathsf{w}}(\{Y \mid X \leq_{\mathsf{T}} Y\})\colon \mathcal{D}_{\mathsf{T}} \to \mathcal{D}_{\mathsf{w}}$.

There is a not-so-obvious embedding of $\mathcal{E}_{\mathsf{T}}$ into $\mathcal{E}_{\mathsf{w}}$, namely $\mathbf{x} \mapsto \mathbf{x} \vee \mathbf{1}\colon \mathcal{E}_{\mathsf{T}} \to \mathcal{E}_{\mathsf{w}}$.

These embeddings are one-to-one and preserve the algebraic structure of $\mathcal{D}_{\mathsf{T}}$ and $\mathcal{E}_{\mathsf{T}}$ respectively. In particular, we have $\mathbf{0} < \mathbf{1} < \mathbf{0}'$.

Examples of Muchnik degrees in $\mathcal{E}_{\mathsf{W}}$.

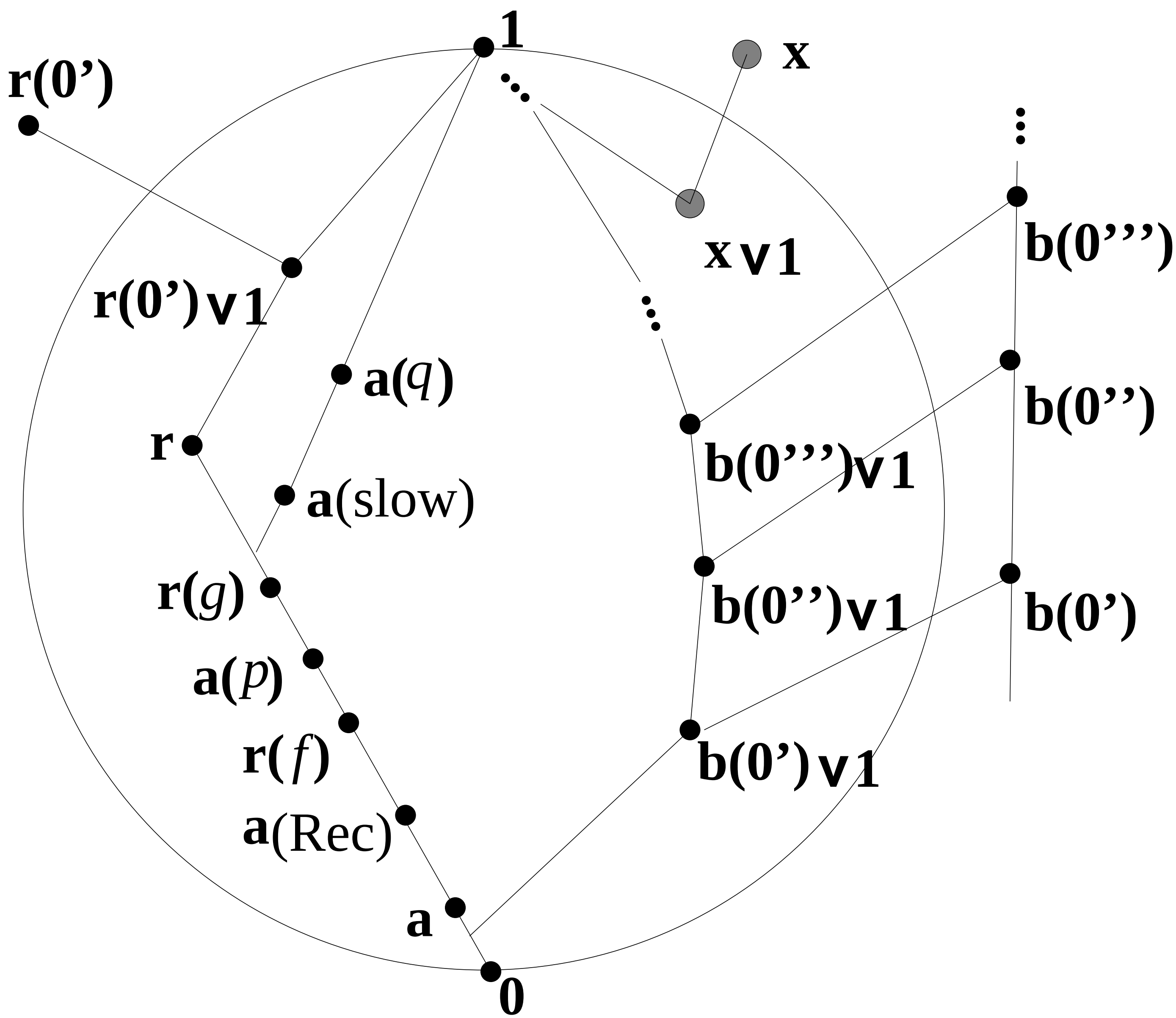
The bottom and top degrees of $\mathcal{E}_{\mathsf{W}}$ are $\mathbf{0}$ and $\mathbf{1}$.
And $\mathbf{r} = \mathbf{r}(\mathbf{0}) \in \mathcal{E}_{\mathsf{W}}$, and $\mathbf{0} < \mathbf{r} < \mathbf{1}$. This degree $\mathbf{r} = \deg_{\mathsf{W}}(\mathsf{MLR})$ is the first non-trivial example of a specific, natural, Muchnik degree in $\mathcal{E}_{\mathsf{W}}$.

For all Muchnik degrees $\mathbf{p}, \mathbf{q} \in \mathcal{D}_{\mathsf{W}}$, we write
$\mathbf{p} \vee \mathbf{q} =$ the greatest lower bound of $\mathbf{p}$ and $\mathbf{q}$, and
$\mathbf{p} \wedge \mathbf{q} =$ the least upper bound of $\mathbf{p}$ and $\mathbf{q}$.
(This is the opposite of the usual lattice notation.)

Remarkably, there are many specific, natural degrees $\mathbf{p} \in \mathcal{D}_{\mathsf{W}}$ such that $\mathbf{p} \notin \mathcal{E}_{\mathsf{W}}$ but $\mathbf{p} \vee \mathbf{1} \in \mathcal{E}_{\mathsf{W}}$.
E.g., for all α in the range $\mathbf{0} < \alpha < \omega_1^{\mathsf{CK}}$,
we have $\mathbf{b}(\mathbf{0}^{(\alpha)}) \notin \mathcal{E}_{\mathsf{W}}$ but $\mathbf{b}(\mathbf{0}^{(\alpha)}) \vee \mathbf{1} \in \mathcal{E}_{\mathsf{W}}$.
And, $\mathbf{r}(\mathbf{0}') \notin \mathcal{E}_{\mathsf{W}}$ but $\mathbf{r}(\mathbf{0}') \vee \mathbf{1} \in \mathcal{E}_{\mathsf{W}}$.
On the other hand, $\mathbf{r}(\mathbf{0}^{(\alpha)}) \vee \mathbf{1} \notin \mathcal{E}_{\mathsf{W}}$ for all $\alpha \geq 2$.

In addition, for all recursively enumerable Turing degrees $\mathbf{x} \in \mathcal{E}_{\mathsf{T}}$, we have $\mathbf{x} \vee \mathbf{1} \in \mathcal{E}_{\mathsf{W}}$. And this gives a specific, natural embedding $\mathbf{x} \mapsto \mathbf{x} \vee \mathbf{1} \colon \mathcal{E}_{\mathsf{T}} \to \mathcal{E}_{\mathsf{W}}$ which preserves the algebraic structure: order, semilattice, top, bottom. However, none of these degrees $\mathbf{x} \vee \mathbf{1}$ are known to be specific and natural, except $\mathbf{0}$ and $\mathbf{1}$.

A picture of $\mathcal{E}_\mathrm{w}$, the lattice of Muchnik degrees of nonempty Π^0_1 subclasses of $\{0,1\}^{\mathbb{N}}$.



Structural analogies between $\mathcal{E}_\mathsf{T}$ and $\mathcal{E}_\mathsf{w}$.

The most famous structural results for $\mathcal{E}_\mathsf{T}$ are the Splitting Theorem and the Density Theorem:

Splitting Theorem for $\mathcal{E}_\mathsf{T}$ (Sacks 1962)**:**
$\mathcal{E}_\mathsf{T}$ satisfies $\forall x\,(x > 0 \Rightarrow \exists u\,\exists v\,(x > u$ and $x > v$ and $x = u \wedge v))$.

Density Theorem for $\mathcal{E}_\mathsf{T}$ (Sacks 1964)**:**
$\mathcal{E}_\mathsf{T}$ satisfies $\forall x\,\forall y\,(x > y \Rightarrow \exists z\,(x > z > y))$.

There are analogous structural results for $\mathcal{E}_\mathsf{w}$:

Splitting Theorem for $\mathcal{E}_\mathsf{w}$ (Binns 2003)**:**
$\mathcal{E}_\mathsf{w}$ satisfies $\forall x\,(x > 0 \Rightarrow \exists u\,\exists v\,(x > u$ and $x > v$ and $x = u \wedge v))$.

Density Theorem for $\mathcal{E}_\mathsf{w}$ (Binns/Shore/Simpson 2016)**:**
$\mathcal{E}_\mathsf{w}$ satisfies $\forall x\,\forall y\,(x > y \Rightarrow \exists z\,(x > z > y))$.

What about **Dense Splitting**?
$\forall x\,\forall y\,(x > y \Rightarrow \exists u\,\exists v\,(x > u > y$ and $x > v > y$ and $x = u \wedge v))$?

We know that $\mathcal{E}_\mathsf{T}$ <u>does not</u> satisfy this. (Lachlan, 1976).

We <u>do not know</u> whether $\mathcal{E}_\mathsf{w}$ satisfies this.

More examples of specific, natural degrees in $\mathcal{E}_{\mathsf{w}}$.

For $f\colon \mathbb{N} \to [0,\infty)$ we define $\mathbf{r}(f) = \deg_{\mathsf{w}}(\mathsf{K}(f))$ where $\mathsf{K}(f) = \{X \in \{0,1\}^{\mathbb{N}} \mid \mathsf{K}(X{\upharpoonright}\{0,1,\ldots,n-1\}) \geq^{+} f(n)\}$. Here K denotes a priori Kolmogorov complexity, and $\geq^{+}$ denotes $\geq$ modulo an additive constant. (Instead of K we could use Martin-Löf-style tests.)

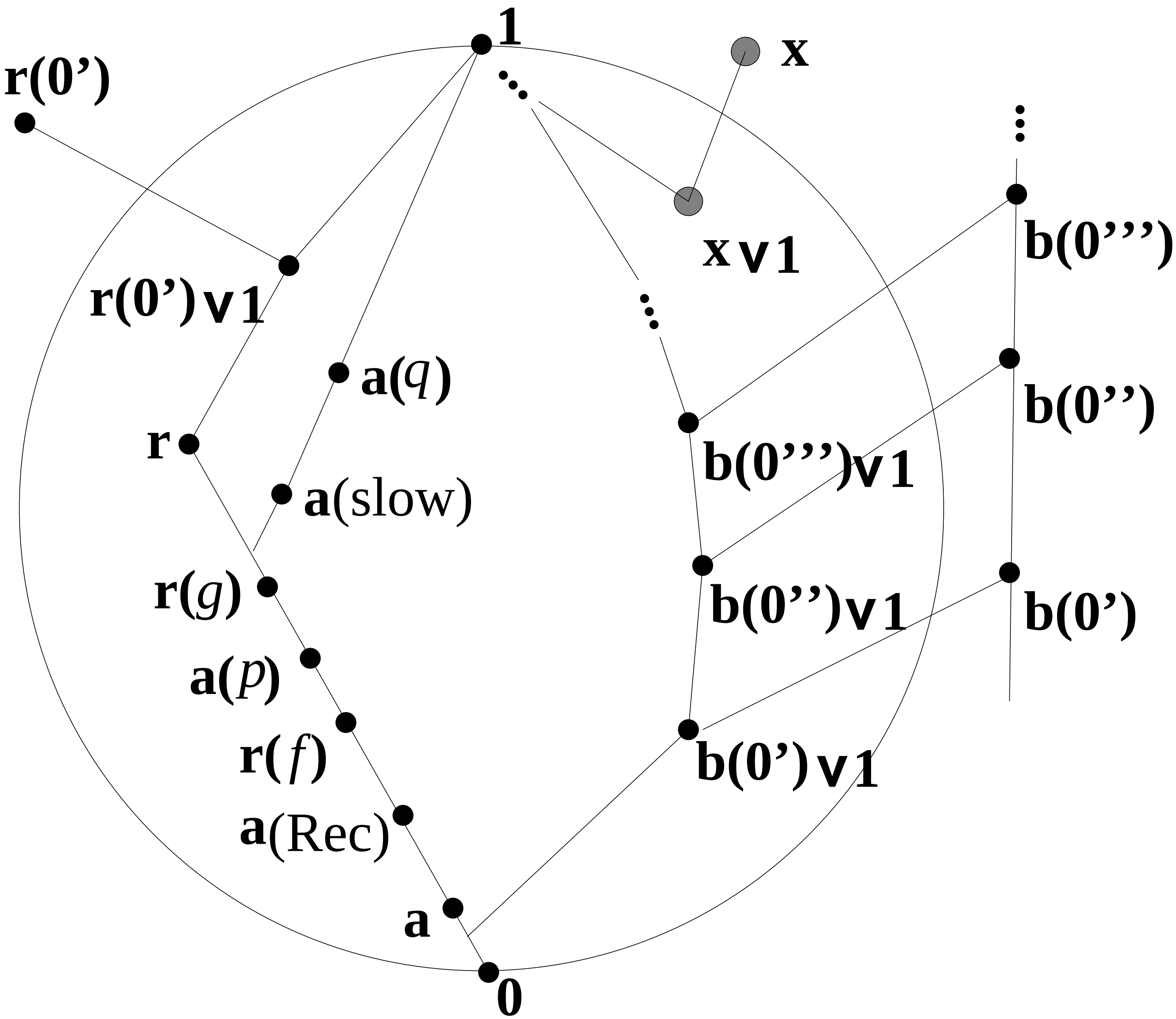
By varying the growth rate of f, we get a hierarchy of specific, natural degrees $\mathbf{r}(f) \in \mathcal{E}_{\mathsf{w}}$. For instance, $\mathbf{r}(\cdot/2) > \mathbf{r}(\cdot/3)$ and $\mathbf{r}(\sqrt[2]{\cdot}) > \mathbf{r}(\sqrt[3]{\cdot})$ and $\mathbf{r}(\log_2 \cdot) > \mathbf{r}(\log_3 \cdot)$. This is called the partial randomness hierarchy.

A partial recursive function $\psi :\subseteq \mathbb{N} \to \mathbb{N}$ is universal if for all partial recursive functions $\varphi :\subseteq \mathbb{N} \to \mathbb{N}$ there exists a recursive function $r\colon \mathbb{N} \to \mathbb{N}$ such that $\varphi(n) \simeq \psi(r(n))$ for all n. And ψ is linearly universal if it is universal via linear functions.

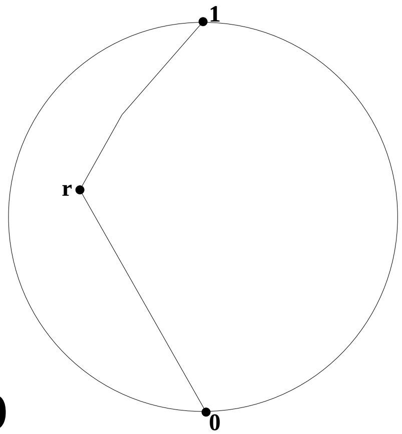
We say that $X \in \mathbb{N}^{\mathbb{N}}$ avoids ψ if $X(n) \not\simeq \psi(n)$ for all n.
Let $\mathbf{a} = \deg_{\mathsf{w}}(\mathsf{UA}) = \deg_{\mathsf{w}}(\mathsf{LUA})$ where $\mathsf{UA} = \{X \in \mathbb{N}^{\mathbb{N}} \mid X$ avoids some universal $\psi\}$ and $\mathsf{LUA} = \{X \in \mathbb{N}^{\mathbb{N}} \mid X$ avoids some linearly universal $\psi\}$.
For $p\colon \mathbb{N} \to [0,\infty)$ we define $\mathbf{a}(p) = \deg_{\mathsf{w}}(\{X \in \mathsf{LUA} \mid X$ is p-bounded$\})$.
This is called the avoidance hierarchy.

We define $\mathbf{a}(\mathsf{Rec}) = \deg_{\mathsf{w}}(\{X \in \mathsf{LUA} \mid X$ is recursively bounded$\})$, and $\mathbf{a}(\mathsf{slow}) = \deg_{\mathsf{w}}(\{X \in \mathsf{LUA} \mid X$ is q-bounded for some slow-growing recursive $q\})$. Here slow-growing means that $\sum_n 1/q(n) = \infty$.

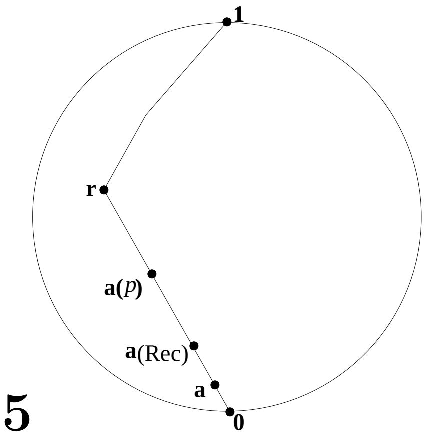
A picture of $\mathcal{E}_W$.



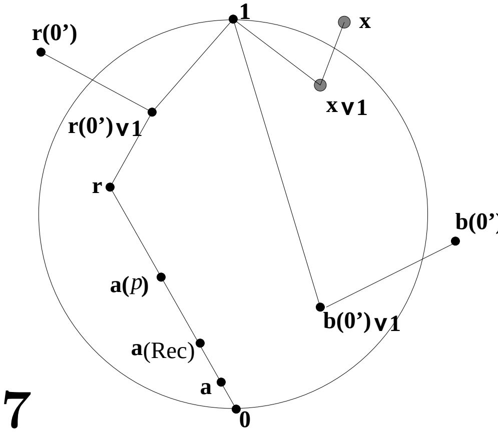
A history of $\mathcal{E}_W$ in pictures.



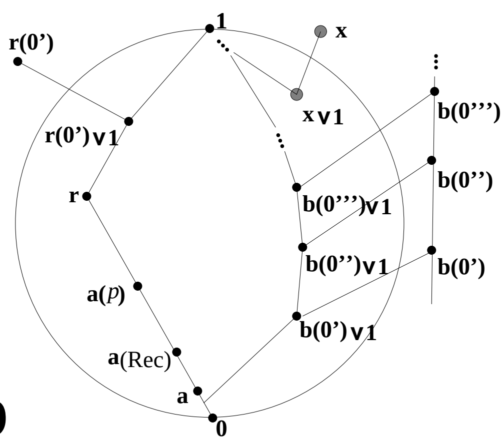
1999



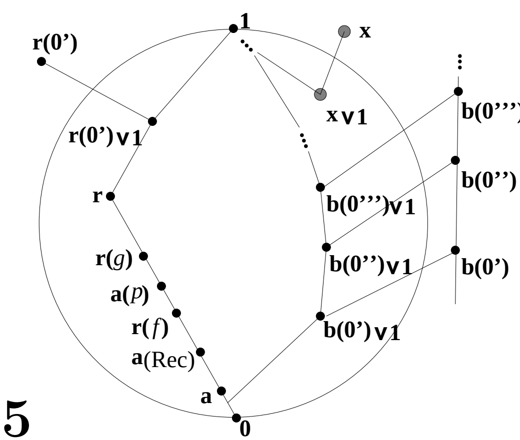
2005



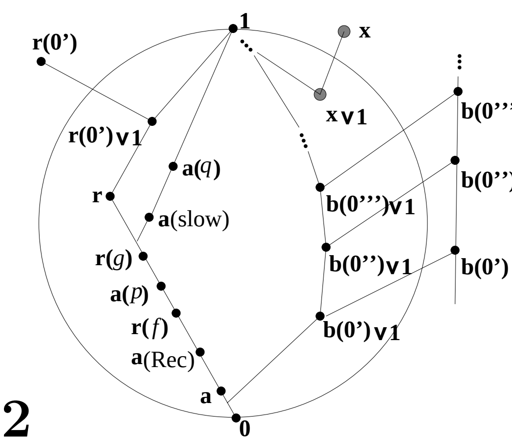
2007



2009



2015



2022

Selected papers related to $\mathcal{E}_\text{w}$ (in chronological order).

Stephen G. Simpson, Natural r.e. degrees and Pi01 classes, FOM e-mail list, August 13, 1999.

Stephen Binns, The Medvedev and Muchnik Lattices of Π^0_1 Classes, Pennsylvania State University, 2003, VII + 80 pages, http://etda.libraries.psu.edu/paper/6092/.

Stephen Binns, A splitting theorem for the Medvedev and Muchnik lattices, Mathematical Logic Quarterly, 2003, 49, 327–335.

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Joshua A. Cole and Stephen G. Simpson, Mass problems and hyperarithmeticity, Journal of Mathematical Logic, 2008, 7, 125–143.

Stephen G. Simpson, Mass problems and measure-theoretic regularity, Bulletin of Symbolic Logic, 2009, 15, 385–409.

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Selected papers related to $\mathcal{E}_{\mathrm{w}}$ (continued).

Joseph S. Miller, Extracting information is hard: a Turing degree of non-integral effective Hausdorff dimension, Advances in Mathematics, 2011, 226, 373–384.

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W. M. Phillip Hudelson, Mass problems and initial segment complexity, Journal of Symbolic Logic, 2014, 79, 20–44.

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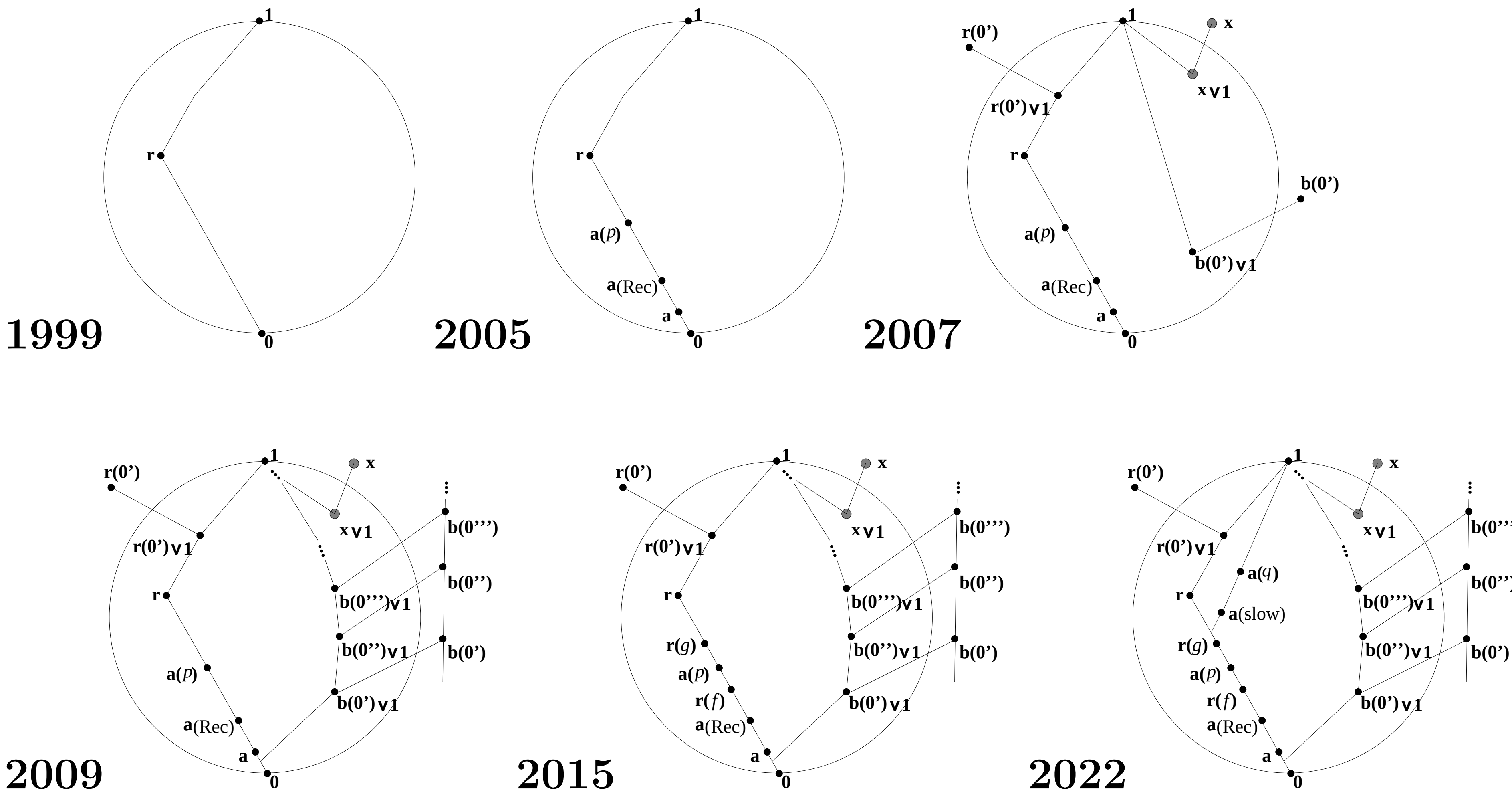
A history of $\mathcal{E}_W$ in pictures.



Thank you for your attention!