DEGREES OF UNSOLVABILITY AND SYMBOLIC DYNAMICS

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Degrees of unsolvability.

Let $\mathbb{N} = \{\text{the natural numbers}\} = \{1, 2, \ldots\}$.
Let $\Omega = \{0, 1\}^{\mathbb{N}} = \{x \mid x : \mathbb{N} \to \{0, 1\}\}$
= the Cantor space.

Consider a Turing machine M with three infinite tapes: the input tape, the output tape, and the scratch tape. Assume that the input tape is read-only and the output tape is write-once. We use M to define a partial functional $\Phi_M :\subseteq \Omega \to \Omega$, as follows.

Given $x, y \in \Omega$ let $M(x) =$ the run of M starting with x on the input tape and blanks on the output and scratch tapes. We define $\Phi_M(x) = y$ if and only if $M(x)$ writes y on the output tape. Otherwise $\Phi_M(x)$ is undefined.

Here x is used as an "oracle" which helps us to compute y. We say that y is *computable relative to* x. This idea came from Turing.

Note that Φ_M is continuous on its domain.

For sets $P, Q \subseteq \Omega$ we define:

$P \geq_{\mathsf{s}} Q$, i.e., Q is *strongly reducible* to P, if and only if $\exists M \, \forall x \, (x \in P \Rightarrow \Phi_M(x) \in Q)$.

$P \geq_{\mathsf{w}} Q$, i.e., Q is *weakly reducible* to P, if and only if $\forall x \, \exists M \, (x \in P \Rightarrow \Phi_M(x) \in Q)$.

Motivation: The sets $P, Q \subseteq \Omega$ are regarded as "problems." The "solutions" of P are the elements of P. Such problems are known as *mass problems*. The problem P is said to be "solvable" if at least one of its solutions is computable. Otherwise P is said to be "unsolvable." The problem Q is said to be "reducible" to the problem P if each solution x of P can be used as a Turing oracle to compute some solution y of Q.

The distinction between $\geq_{\mathsf{s}}$ and $\geq_{\mathsf{w}}$ lies in whether or not the Turing machine M which computes y relative to x is required to be independent of x.

History:

Kolmogorov 1932 developed his "calculus of problems" as a nonrigorous yet compelling explanation of Brouwer's intuitionism. Later Medvedev 1955 and Muchnik 1963 proposed strong and weak reducibility as rigorous explications of Kolmogorov's idea.

Some references:

Stephen G. Simpson, Mass problems and randomness, Bulletin of Symbolic Logic, 11, 2005, pages 1–27.

Stephen G. Simpson, Medvedev degrees of 2-dimensional subshifts of finite type, 8 pages, 1 May 2007; accepted 26 September 2007 for publication in Ergodic Theory and Dynamical Systems.

Stephen G. Simpson, Mass problems and intuitionism, Notre Dame Journal of Formal Logic, 49, 2008, pages 127–136.

Stephen G. Simpson, Mass problems and measure-theoretic regularity, Bulletin of Symbolic Logic, 15, 2009, pages 385–409.

Stephen G. Simpson, Entropy equals dimension equals complexity, 2010, in preparation.

More definitions:

$P \equiv_{\mathsf{s}} Q \Leftrightarrow (P \leq_{\mathsf{s}} Q \wedge Q \leq_{\mathsf{s}} P)$.

$P \equiv_{\mathsf{w}} Q \Leftrightarrow (P \leq_{\mathsf{w}} Q \wedge Q \leq_{\mathsf{w}} P)$.

$$\begin{aligned}\deg_{\mathsf{s}}(P) &= \{Q \mid P \equiv_{\mathsf{s}} Q\} \\ &= \text{the } \textit{strong degree} \text{ of } P.\end{aligned}$$

$$\begin{aligned}\deg_{\mathsf{w}}(P) &= \{Q \mid P \equiv_{\mathsf{w}} Q\} \\ &= \text{the } \textit{weak degree} \text{ of } P.\end{aligned}$$

$\mathcal{D}_{\mathsf{s}} = \{\deg_{\mathsf{s}}(P) \mid P \subseteq \Omega\}$.

$\mathcal{D}_{\mathsf{s}}$ has a partial ordering $\leq$ induced by $\leq_{\mathsf{s}}$.

$\mathcal{D}_{\mathsf{w}} = \{\deg_{\mathsf{w}}(P) \mid P \subseteq \Omega\}$.

$\mathcal{D}_{\mathsf{w}}$ has a partial ordering $\leq$ induced by $\leq_{\mathsf{w}}$.

Remark. Medvedev 1955 and Muchnik 1963 respectively noted that $\mathcal{D}_{\mathsf{s}}$ and $\mathcal{D}_{\mathsf{w}}$ are complete Brouwerian lattices. Aspects of these lattices have been studied by Dyment, Skvortsova, Sorbi, Terwijn, and others.

Note. The cardinality of $\mathcal{D}_{\mathsf{s}}$ and $\mathcal{D}_{\mathsf{w}}$ is $2^{2^{\aleph_0}}$.

Yet more definitions:

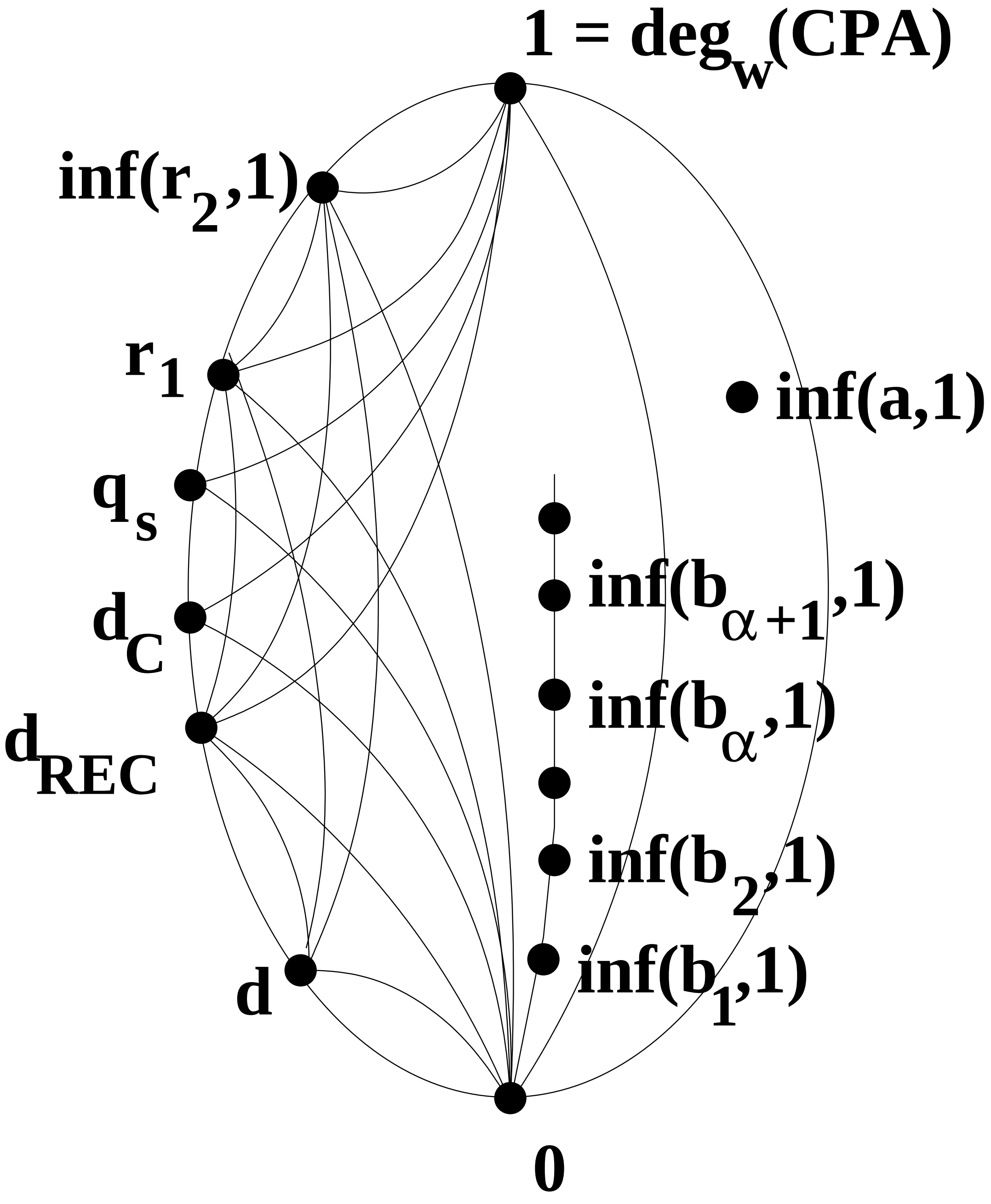
$U \subseteq \Omega$ is *effectively open* if $U = \bigcup_{n=1}^{\infty} \Omega_{g(n)}$ for some comptable function $g : \mathbb{N} \to \{0,1\}^*$.

$P \subseteq \Omega$ is *effectively closed* if its complement $\Omega \setminus P$ is effectively open.

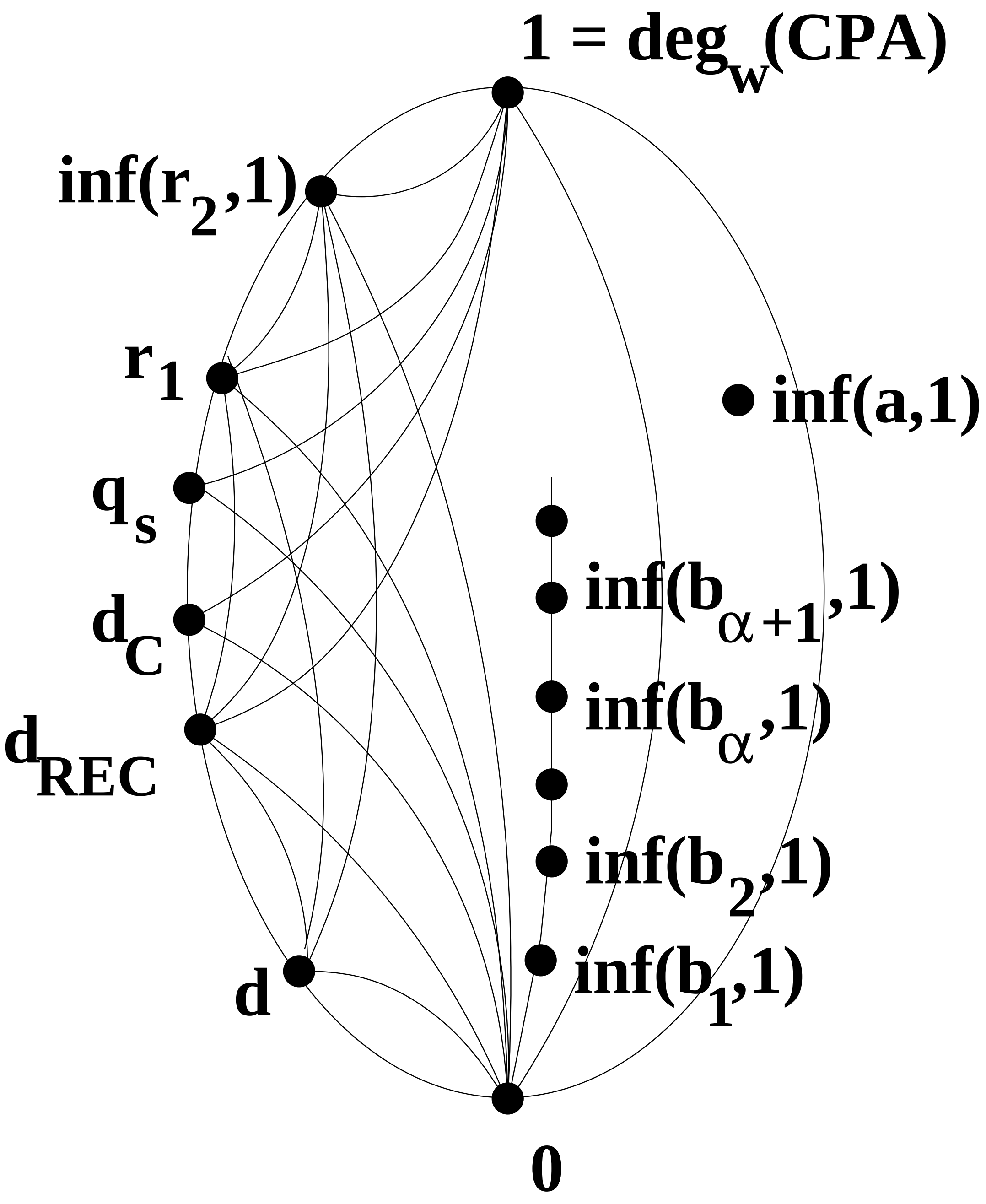
$\mathcal{E}_\mathsf{s} = \{\deg_\mathsf{s}(P) \mid \emptyset \neq P \subseteq \Omega,\ P \text{ eff. closed}\}$.

$\mathcal{E}_\mathsf{w} = \{\deg_\mathsf{w}(P) \mid \emptyset \neq P \subseteq \Omega,\ P \text{ eff. closed}\}$.

Remark. $\mathcal{E}_\mathsf{s}$ and $\mathcal{E}_\mathsf{w}$ are countable sublattices of $\mathcal{D}_\mathsf{s}$ and $\mathcal{D}_\mathsf{w}$ respectively. Much is known about them. For instance, both $\mathcal{E}_\mathsf{s}$ and $\mathcal{E}_\mathsf{w}$ contain a bottom degree, denoted $\mathbf{0}$, and a top degree, denoted $\mathbf{1}$. Obviously $\mathbf{0} = \deg_\mathsf{s}(\Omega) = \deg_\mathsf{w}(\Omega)$. However, the existence of $\mathbf{1}$ in $\mathcal{E}_\mathsf{s}$ and $\mathcal{E}_\mathsf{w}$ is not so obvious. A well-known characterization of $\mathbf{1}$ will be mentioned later.



A picture of $\mathcal{E}_w$. Each black dot except $\inf(\mathbf{a}, \mathbf{1})$ represents a specific, natural degree in $\mathcal{E}_w$. We shall explain some of these degrees.



A picture of $\mathcal{E}_{\mathrm{w}}$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{r} =$ randomness, $\mathbf{b} =$ LR-reducibility, $\mathbf{q} =$ dimension, $\mathbf{d} =$ diagonal nonrecursiveness.

We now explain some degrees in $\mathcal{E}_\mathsf{w}$.

The top degree in $\mathcal{E}_\mathsf{w}$ is $\mathbf{1} = \deg_\mathsf{w}(\mathsf{CPA})$ where CPA is the problem of finding a complete, consistent theory which includes first-order arithmetic.

We also have $\inf(\mathbf{a}, \mathbf{1}) \in \mathcal{E}_\mathsf{w}$ where $\mathbf{a}$ is any recursively enumerable Turing degree. Moreover, $\mathbf{a} < \mathbf{b}$ implies $\inf(\mathbf{a}, \mathbf{1}) < \inf(\mathbf{b}, \mathbf{1})$

We have $\mathbf{r}_1 \in \mathcal{E}_\mathsf{w}$ where $\mathbf{r}_1 = \deg_\mathsf{w}(\{z \in \Omega \mid$ z is random in the sense of Martin-Löf$\})$.

We also have $\inf(\mathbf{r}_2, \mathbf{1}) \in \mathcal{E}_\mathsf{w}$ where $\mathbf{r}_2 = \deg_\mathsf{w}(\{z \in \Omega \mid z$ is 2-*random*$\})$, i.e., random relative to the halting problem.

Also $\mathbf{d} \in \mathcal{E}_\mathsf{w}$ where $\mathbf{d} = \deg_\mathsf{w}(\{f \in \mathbb{N}^\mathbb{N} \mid$ f is *diagonally nonrecursive*$\})$, i.e., $\forall n\,(f(n) \neq \varphi_n(n))$.

Let REC $= \{g \in \mathbb{N}^{\mathbb{N}} \mid g$ is recursive$\}$.
Let C be any "nice" subclass of REC.
For instance $C =$ REC, or $C = \{g \in$ REC $\mid$ g is primitive recursive$\}$. We have $\mathbf{d}_C \in \mathcal{E}_\mathsf{w}$ where $\mathbf{d}_C = \deg_\mathsf{w}(\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is diagonally nonrecursive and C-*bounded*$\})$,
i.e., $(\exists g \in C)\,\forall n\,(f(n) < g(n))$.

Also, $\mathbf{d}_C = \deg_\mathsf{w}(\{z \in \Omega \mid z$ is C-*complex*$\}$,
i.e., $(\exists g \in C)\,\forall n\,(\mathsf{K}(z{\restriction}\{1,\ldots,g(n)\}) \geq n)\})$.
Moreover, $\mathbf{d}_{C'} < \mathbf{d}_C$ whenever C' contains a function which dominates all functions in C.

For $z \in \Omega$ let effdim$(z) =$ the *effective Hausdorff dimension* of z, i.e.,

$$\mathrm{effdim}(z) = \liminf_{n\to\infty} \frac{\mathsf{K}(z{\restriction}\{1,\ldots,n\})}{n}.$$

Given a right recursively enumerable real number $s < 1$, we have $\mathbf{q}_s \in \mathcal{E}_\mathsf{w}$ where

$$\mathbf{q}_s = \deg_\mathsf{w}(\{z \mid \mathrm{effdim}(z) > s\}).$$

Moreover, $s < t$ implies $\mathbf{q}_s < \mathbf{q}_t$ (Miller).

Using x as an oracle, define

$\mathsf{R}^x = \{z \in \Omega \mid z \text{ is random relative to } x\}$

and $\mathsf{K}^x(n)$ = the prefix-free Kolmogorov complexity of n relative to x.

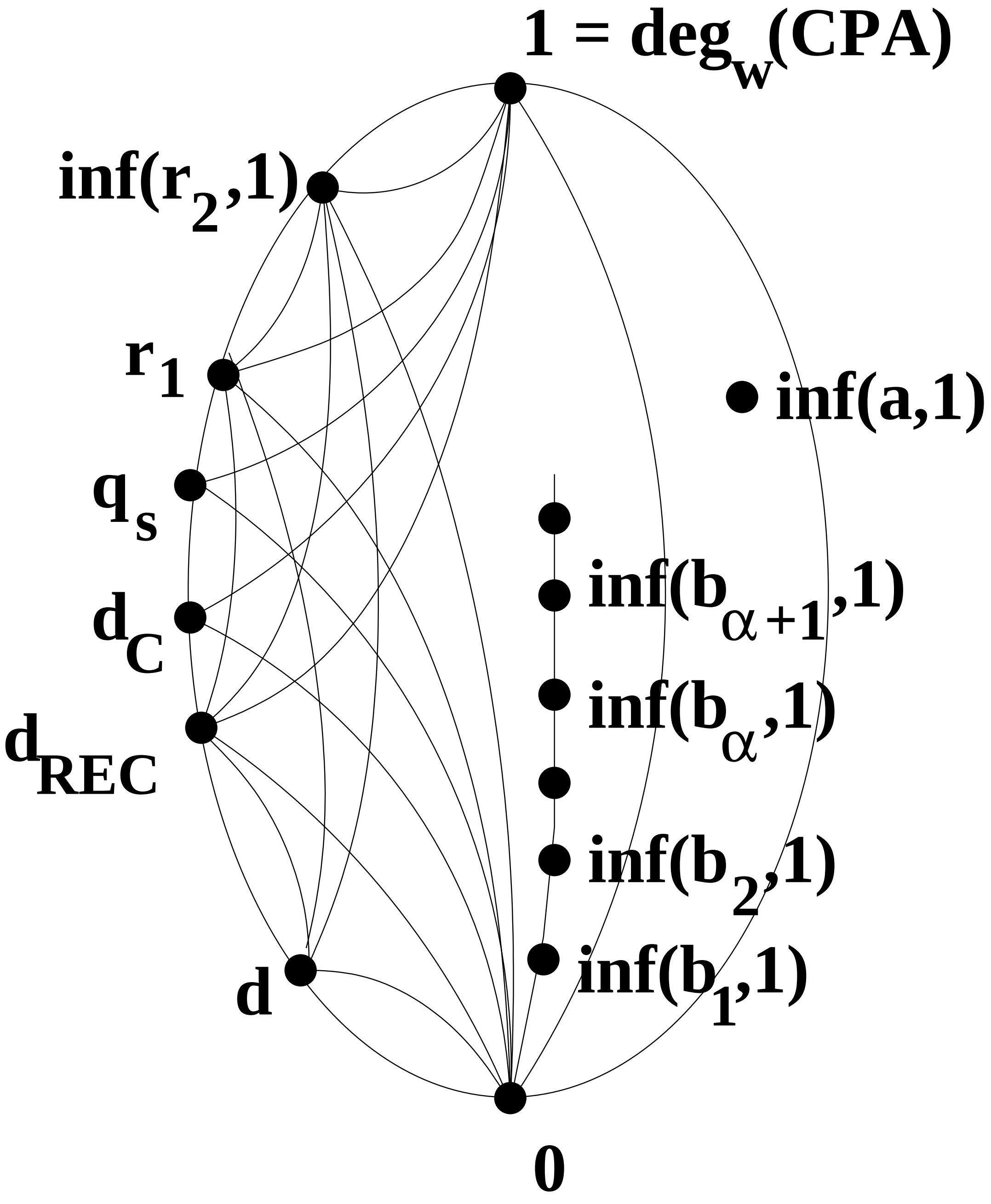
Define $x \leq_{\mathsf{LR}} y \Leftrightarrow \mathsf{R}^y \subseteq \mathsf{R}^x$

and $x \leq_{\mathsf{LK}} y \Leftrightarrow \exists c \forall n\, (\mathsf{K}^y(n) \leq \mathsf{K}^x(n) + c)$.

Theorem (Miller/Kjos-Hanssen/Solomon)**.** We have $x \leq_{\mathsf{LR}} y$ if and only if $x \leq_{\mathsf{LK}} y$.

For each recursive ordinal number α, let $0^{(\alpha)}$ = the αth iterated Turing jump of 0. Thus $0^{(1)}$ = the halting problem, and $0^{(\alpha+1)}$ = the halting problem relative to $0^{(\alpha)}$, etc. This is the hyperarithmetical hierarchy. We embed it naturally into $\mathcal{E}_{\mathsf{w}}$ as follows.

Theorem (Simpson 2009)**.** $0^{(\alpha)} \leq_{\mathsf{LR}} y \Leftrightarrow$ every $\Sigma^0_{\alpha+2}$ set includes a $\Sigma^{0,y}_2$ set of the same measure. Moreover, letting $\mathbf{b}_\alpha = \deg_{\mathsf{w}}(\{y \mid 0^{(\alpha)} \leq_{\mathsf{LR}} y\})$ we have $\inf(\mathbf{b}_\alpha, \mathbf{1}) \in \mathcal{E}_{\mathsf{w}}$ and $\inf(\mathbf{b}_\alpha, \mathbf{1}) < \inf(\mathbf{b}_{\alpha+1}, \mathbf{1})$.



A picture of $\mathcal{E}_{\mathrm{w}}$. Here $\mathbf{a}$ = any r.e. degree, $\mathbf{r}$ = randomness, $\mathbf{b}$ = LR-reducibility, $\mathbf{q}$ = dimension, $\mathbf{d}$ = diagonal nonrecursiveness.

Symbolic dynamics.

Let A be a finite set of symbols. Let
$\mathbb{Z}$ = the integers = $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
We write $A^{\mathbb{Z}} = \{x \mid x : \mathbb{Z} \to A\}$.
This is the *full shift space* on A.
The *shift operator* $S : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is given by $S(x)(i) = x(i+1)$ for all $i \in \mathbb{Z}$.

A *subshift* is a set $X \subseteq A^{\mathbb{Z}}$ which is nonempty, closed, and *shift invariant*, i.e., $\forall x\,(x \in X \Leftrightarrow S(x) \in X)$.

If X and Y are subshifts, a *shift morphism* from X to Y is a continuous mapping $f : X \to Y$ such that $f(S(x)) = S(f(x))$ for all $x \in X$.

Symbolic dynamics is the study of subshifts and shift morphisms.

Subshifts defined by excluded words.

Given $E \subseteq A^* = \bigcup_{n=1}^{\infty} A^n$ let $X_E =$

$\{x \in A^{\mathbb{Z}} \mid (\forall i \in \mathbb{Z})\, \forall n\, \langle x(i+1), \ldots, x(i+n) \rangle \notin E\}$.

Thus E is a set of “excluded words.”

Clearly X_E is a subshift, provided it is $\neq \emptyset$.

Moreover, all subshifts are of this form.

If E is finite, the subshift X_E
is said to be *of finite type*.

If E is computable, the subshift X_E
is said to be *of computable type*.

Some easy remarks:

1. If $f : X \to Y$ is a shift morphism, then $X \geq_{\mathsf{s}} Y$ and $X \geq_{\mathsf{w}} Y$.

   In fact, f is a "block code."

2. If $f, f^{-1} : X \leftrightarrow Y$ are shift morphisms, then $X \equiv_{\mathsf{s}} Y$ and $X \equiv_{\mathsf{w}} Y$.

   Thus, the strong and weak degrees of a subshift are "conjugacy invariants."

3. X is of computable type if and only if X is effectively closed.

4. If X is of computable type, then $\deg_{\mathsf{s}}(X) \in \mathcal{E}_{\mathsf{s}}$ and $\deg_{\mathsf{w}}(X) \in \mathcal{E}_{\mathsf{w}}$.

Theorem (Joseph Miller)**.** Conversely, each degree in $\mathcal{E}_{\mathsf{s}}$ or $\mathcal{E}_{\mathsf{w}}$ is, respectively, the strong or weak degree of a subshift of computable type.

The proof is ingenious but not difficult.

We now generalize to d-dimensional subshifts. For $d \geq 1$ we write $\mathbb{Z}^d = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{d}$.

As before, let A be a finite set of symbols. The *full d-dimensional shift space* over A is $A^{\mathbb{Z}^d} = \{x \mid x : \mathbb{Z}^d \to A\}$. The *shift operators* $S_k : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ for $k = 1, \ldots, d$ are given by $S_k(x)(i_1, \ldots, i_d) = x(i_1, \ldots, i_k + 1, \ldots, i_d)$.

A *d-dimensional subshift* is a set $X \subseteq A^{\mathbb{Z}^d}$ which is closed, nonempty, and *shift invariant*, i.e., $(\forall k)_{1 \leq k \leq d} \forall x\, (x \in X \Leftrightarrow S_k(x) \in X)$.

Each d-dimensional subshift is of the form $X_E = \{x \in A^{\mathbb{Z}^d} \mid \forall n\, (\forall i_1, \ldots, i_d \in \mathbb{Z})$ $(\langle x(i_1 + j_1, \ldots, i_d + j_d)\rangle_{1 \leq j_1, \ldots, j_d \leq n} \notin E)\}$

where $E \subseteq \bigcup_{n=1}^{\infty} A^{\{1,\ldots,n\}^d}$. Thus E is a set of "excluded d-dimensional words."

If E is finite, we say that X_E is *<u>of finite type</u>*.

If E is computable, we say that X_E is *<u>of computable type</u>*.

All of our earlier remarks concerning the 1-dimensional case extend easily to the d-dimensional case.

Theorem (Simpson)**.** Each degree in $\mathcal{E}_{\mathrm{S}}$ or $\mathcal{E}_{\mathrm{W}}$ is, respectively, the strong or weak degree of a 2-dimensional subshift of <u>finite</u> type.

The proof uses techniques going back to R. Berger 1965 and R. Robinson 1972. Another proof can be obtained by means of "self-replicating tile sets" (Durand/Romashchenko/Shen).

Remark. There are many specific, interesting degrees in $\mathcal{E}_{\mathrm{W}}$. By the above theorems, each degree in $\mathcal{E}_{\mathrm{W}}$ is realized by

(a) a <u>1-dimensional</u> subshift of <u>computable</u> type (Miller), and

(b) a <u>2-dimensional</u> subshift of <u>finite</u> type (Simpson).

A possibly interesting research program:

Given a subshift X, explore the relationship between the <u>*dynamical properties*</u> of X and the <u>*degree of unsolvability*</u> of X, i.e., $\deg_{\mathsf{s}}(X)$ or $\deg_{\mathsf{w}}(X)$.

For example, the *entropy* of X is a well-known dynamical property which serves as an <u>upper bound</u> on the complexity of orbits. In particular $\mathrm{ent}(X) > 0$ implies $(\exists x \in X)\,(x$ is not computable).

By contrast, the degree of unsolvability of X serves as a <u>lower bound</u> on the complexity of orbits. E.g., $\deg_{\mathsf{s}}(X) > \mathbf{0} \Leftrightarrow \deg_{\mathsf{w}}(X) > \mathbf{0} \Leftrightarrow$ $(\forall x \in X)\,(x$ is not computable).

Theorem (Hochman)**.** If X is of computable type and *minimal* (i.e., every orbit is dense), then $\deg_{\mathsf{s}}(X) = \deg_{\mathsf{w}}(X) = \mathbf{0}$.

The proof is not difficult.

Some new (?) results on subshifts:

Let d be a positive integer, let A be a finite set of symbols, and let X be a subset of $A^{\mathbb{N}^d}$ (the "one-sided" case) or of $A^{\mathbb{Z}^d}$ (the "two-sided" case).

The *Hausdorff dimension*, $\dim(X)$, and the *effective Hausdorff dimension*, $\mathrm{effdim}(X)$, are defined as usual with respect to the standard metric $\rho(x,y) = 2^{-|F_n|}$ where n is as large as possible such that $x{\upharpoonright}F_n = y{\upharpoonright}F_n$. Here $F_n = \{1,\ldots,n\}^d$ in the one-sided case, and $F_n = \{-n,\ldots,n\}^d$ in the two-sided case.

We first state some old results.

1. $\mathrm{effdim}(X) = \sup\limits_{x \in X} \mathrm{effdim}(\{x\})$.

2. $\mathrm{effdim}(\{x\}) = \liminf\limits_{n\to\infty} \dfrac{\mathsf{K}(x{\upharpoonright}F_n)}{|F_n|}$.

3. $\mathrm{effdim}(X) = \dim(X)$

   provided X is <u>effectively closed</u>.

We now state some apparently new results.

Theorem (2010)**.** Assume that X is nonempty, closed and shift-invariant. Then

$$\mathsf{effdim}(X) = \dim(X) = \mathsf{ent}(X).$$

Moreover,

$$\dim(X) \geq \limsup_{n\to\infty} \frac{\mathsf{K}(x{\restriction}F_n)}{|F_n|} \quad \text{for all } x \in X,$$

and

$$\dim(X) = \lim_{n\to\infty} \frac{\mathsf{K}(x{\restriction}F_n)}{|F_n|} \quad \text{for some } x \in X.$$

Note. Here X can be any kind of subshift: 1-sided or 2-sided, effectively closed or closed, 1-dimensional or d-dimensional.

Remark. Here $\mathsf{ent}(X)$ denotes *entropy*,

$$\mathsf{ent}(X) = \lim_{n\to\infty} \frac{\log_2 |\{x{\restriction}F_n \mid x \in X\}|}{|F_n|}.$$

This is known to be a conjugacy invariant.

Remark. The proof of this theorem involves ergodic theory (Shannon/McMillan/Breiman, the Variational Principle, etc.) plus a combinatorial argument which is similar to the proof of the Vitali Covering Lemma.

Remark. So far as I can tell, everything in the theorem is new, except the following old result due to Furstenberg: $\text{dim}(X) = \text{ent}(X)$ provided X is one-sided and 1-dimensional. The proof of this special case is much easier.

Remark. The above theorem is an outcome of my discussions at Penn State over the past several months with many people including John Clemens, Michael Hochman, Daniel Mauldin, Jan Reimann, and Alexander Shen.

THE END.

THANK YOU!