Computable Symbolic Dynamics

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A 1-*dimensional dynamical system* is an ordered pair (Y, T) where $T : Y \to Y$. The elements of the set Y are the *states* of the system, and T is the *state transition function*. Given a state $y \in Y$, the *orbit* or *trajectory* of y is the sequence $T^n y$, $n = 0, 1, 2, \ldots$. One considers the behavior of $T^n y$ as n goes to infinity.

Often one assumes that Y is a compact Polish space and T is a homeomorphism of Y onto Y. Thus for each $y \in Y$ one can consider the biinfinite trajectory $T^n y$, $n \in \mathbb{Z}$.

Given a partition $C_1, \ldots, C_k$ of Y, define $X \subseteq \{1, \ldots, k\}^{\mathbb{Z}}$ by
$X = \{x \mid (\exists y \in Y)\,(\forall n \in \mathbb{Z})\,(T^n y \in C_{x(n)})\}$.
Thus x is the "trace" or "code" of y in the *symbolic system* (X, S). Here $S : X \to X$ is the *shift operator* given by
$(Sx)(n) = x(n + 1)$.

If Y is compact and $C_1, \ldots, C_k$ are closed subsets of Y, then X is a compact subset of $\{1, \ldots, k\}^{\mathbb{Z}}$.

We think of (X, S) as
a symbolic representation of (Y, T).

Thus symbolic dynamical systems are useful in the study of arbitrary dynamical systems.

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All of the concepts above can be generalized, replacing $\mathbb{Z}$ by an arbitrary group G. In this talk we always assume that G is computable.

Many results concern the *d-dimensional* case, $G = \mathbb{Z}^d$. However, some results hold for an arbitrary computable group G.

Let $A = \{a_1, \ldots, a_k\}$ be an *alphabet*, i.e., a finite set of symbols.
Let G be a computable group.
We write $A^G = \{x \mid x : G \to A\}$.

Let $\sigma, \tau, \ldots$ range over functions $\sigma : F \to A$ where $F = \text{dom}(\sigma)$ is a finite subset of G. The set of such functions is denoted A^G_*. For $\sigma \in A^G_*$ let

$$N_\sigma = \{x \in A^G \mid x \upharpoonright \text{dom}(\sigma) = \sigma\}.$$

The N_σ's are a basis for the standard product topology on A^G. Thus $C \subseteq A^G$ is topologically closed if and only if $C = A^G \setminus \bigcup_{\sigma \in D} N_\sigma$ for some $D \subseteq A^G_*$. If D is computable, we say that C is *effectively closed*, i.e., Π^0_1.

The *shift action* S of G on A^G is given by $(S^g x)(h) = x(gh)$ for all $x \in A^G$ and $g, h \in G$.

A G-*subshift* is a set $X \subseteq A^G$ which is nonempty, topologically closed, and closed under the action of G, i.e., $x \in X$ implies $S^g x \in X$.

We write

$$\mathsf{Seq}(x) = \{\sigma \in A_*^G \mid \exists g\,(x^g \restriction \mathsf{dom}(\sigma) = \sigma)\}.$$

This notation is inspired by the book *Ramsey Theory* by Graham, Rothschild and Spencer.

Given $E \subseteq A_*^G$ let

$$X_E = \{x \in A^G \mid \mathsf{Seq}(x) \cap E = \emptyset\}$$

provided this set is nonempty.
Clearly X_E is a subshift.
We say that the subshift X_E is defined by a set of *excluded configurations*, E.

It is known that any subshift is defined by a set of excluded configurations. In other words, given a subshift X we can find $E \subseteq A_*^G$ such that $X = X_E$.

If E is finite, we say that X_E is *of finite type*.

If E is computable, we say that X_E is *of computable type*.

It can be shown that a subshift is of computable type if and only if it is effectively closed.

It is known that most or all subshifts which arise in practice are of computable type. Here is a precise general result.

Theorem 1. Let G act effectively on an effectively closed, effectively totally bounded set Y in an effectively presented complete separable metric space. For each $a \in A$ let C_a be an effectively closed subset of Y. Let

$$E = \{\sigma \mid \neg(\exists y \in Y)\,(\forall g \in \text{dom}(\sigma))\,(S^g y \in C_{x(g)})\}.$$

If $X_E \neq \emptyset$ then X_E is of computable type.

Let $P(X)$ be the problem of finding a point of X. We have the following theorem.

Theorem 2 (Michael Hochman, 2008). If X is of computable type and *minimal* (i.e., every orbit is dense), then $P(X)$ is algorithmically solvable.

Proof. Write $\mathsf{Seq}(X) = \bigcup_{x \in X} \mathsf{Seq}(x)$. Because X is minimal, we have

$$(\forall x, y \in X) \, \forall F \, \exists g \, (x^g \restriction F = y \restriction F),$$

i.e.,

$$(\forall x, y \in X) \, \forall F \, \forall g \, \exists h \, (x^g \restriction F = y^h \restriction F),$$

i.e.,

$$(\forall x, y \in X) \, (\mathsf{Seq}(x) = \mathsf{Seq}(y)).$$

Thus

$$\sigma \in \mathsf{Seq}(X) \Leftrightarrow (\forall x \in X) \, (\sigma \in \mathsf{Seq}(x))$$

$$\Leftrightarrow \forall x \, (x \in X \Rightarrow \exists g \, (x^g \restriction \mathsf{dom}(\sigma) = \sigma))$$

and a Tarski/Kuratowski computation shows that $\mathsf{Seq}(X)$ is Σ^0_1, i.e., it is the range of a computable sequence. On the other hand,

$$\begin{aligned} \sigma \in \mathsf{Seq}(X) & \Leftrightarrow & (\exists x \in X)\,\exists g\,(x^g \restriction \mathsf{dom}(\sigma) = \sigma) \\ & \Leftrightarrow & \exists y\,(y \in X \wedge y \restriction \mathsf{dom}(\sigma) = \sigma) \end{aligned}$$

and a Tarski/Kuratowski computation shows that $\mathsf{Seq}(X)$ is Π^0_1, i.e., it is the complement of a Σ^0_1 set. It follows that $\mathsf{Seq}(X)$ is Δ^0_1, i.e., computable. Now fix a computable sequence of finite sets $\emptyset = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$ with $\bigcup_{n=0}^{\infty} F_n = G$. Starting with $\sigma_0 = \emptyset$ and given $\sigma_n \in \mathsf{Seq}(X)$ with $\mathsf{dom}(\sigma_n) = F_n$, search for $\sigma_{n+1} \in \mathsf{Seq}(X)$ extending σ_n with $\mathsf{dom}(\sigma_{n+1}) = F_{n+1}$. Finally $x = \bigcup_{n=0}^{\infty} \sigma_n$ is a point of X and is computable, Q.E.D.

Remark. Theorems 1 and 2 hold more generally, when G is a recursively presented semigroup with identity.

If $G = \mathbb{Z}^d$ we say that X is d-dimensional.

We now consider 1-dimensional subshifts, i.e., $G = \mathbb{Z}$.

Theorem 3 ("classical"). If X is 1-dimensional of finite type, then X contains periodic points.

Corollary. If X is 1-dimensional and of finite type, then $P(X)$ is algorithmically solvable.

Theorem 4 (Cenzer/Dashti/King, 2006). If X is 1-dimensional of computable type, then $P(X)$ can be algorithmically unsolvable.

Theorem 5 (Joseph S. Miller, 2008). If X is 1-dimensional of computable type, then $P(X)$ can have any desired degree of unsolvability.

This immediately implies the theorem of Cenzer/Dashti/King.

Restatement of Miller's theorem:

Given a nonempty effectively closed set $C \subseteq \{0,1\}^{\mathbb{N}}$, we can find a 1-dimensional subshift $X \subseteq \{0,1\}^{\mathbb{Z}}$ of computable type, plus computable functionals $\Phi : C \to X$ and $\Psi : X \to C$.

Proof of Miller's theorem. We write $\{0,1\}^* = \bigcup_{n=0}^{\infty} \{0,1\}^n = \{$finite strings of 0's and 1's$\}$. For $s \in \{0,1\}^*$ define $a_s, b_s \in \{0,1\}^*$ by induction on the length of s as follows. Start with $a_\emptyset = 0$ and $b_\emptyset = 1$. Given a_s and b_s define

$$a_{s0} = a_s a_s a_s a_s b_s, \quad a_{s1} = a_s a_s a_s b_s b_s,$$

$$b_{s0} = a_s a_s b_s b_s b_s, \quad b_{s1} = a_s b_s b_s b_s b_s$$

and note that a_s is the middle fifth of a_{s0} and a_{s1} while b_s is the middle fifth of b_{s0} and b_{s1}.

Given $C \subseteq \{0,1\}^{\mathbb{N}}$ let $Q_C = \bigcup_{z \in C} Q_z \subseteq \{0,1\}^{\mathbb{Z}}$ where

$$Q_z = \{x \mid \forall n\, (x \text{ is made of } a_{z\restriction n}\text{'s and } b_{z\restriction n}\text{'s})\}.$$

It is straightforward to show that if C is nonempty and effectively closed then Q_C is a subshift of computable type. Moreover, we have computable functionals $\Phi : C \to Q_C$ and $\Psi : Q_C \to C$ given by $\Phi(z) = \bigcup_{n=0}^{\infty} a_{z\restriction n}$ and $\Psi(x) =$ the unique $z \in C$ such that $x \in Q_z$. Thus, letting $X = Q_C$, we have the desired result, Q.E.D.

Remark. For each $z \in \{0,1\}^{\mathbb{N}}$, Q_z is a minimal subshift. Thus Q_C is a dynamical system with the property that the closure of every orbit is minimal. This property of dynamical systems is somewhat unusual.

Now for the 2-dimensional case, $G = \mathbb{Z} \times \mathbb{Z}$.

Theorem 6 (Berger, 1965). If X is 2-dimensional of finite type, then X can be *aperiodic*, i.e., it has no periodic points.

Theorem 7 (Myers, 1974). If X is 2-dimensional of finite type, then $P(X)$ can be algorithmically unsolvable.

Theorem 8 (Simpson, 2007). If X is 2-dimensional of finite type, then $P(X)$ can have any desired degree of unsolvability.

In other words, given a nonempty effectively closed set $C \subseteq \{0,1\}^{\mathbb{N}}$, we can find a 2-dimensional subshift $X \subseteq \{0,1\}^{\mathbb{Z}\times\mathbb{Z}}$ of finite type along with computable functionals $\Phi : C \to X$ and $\Psi : X \to C$.

This immediately implies Myers's theorem, which immediately implies Berger's theorem.

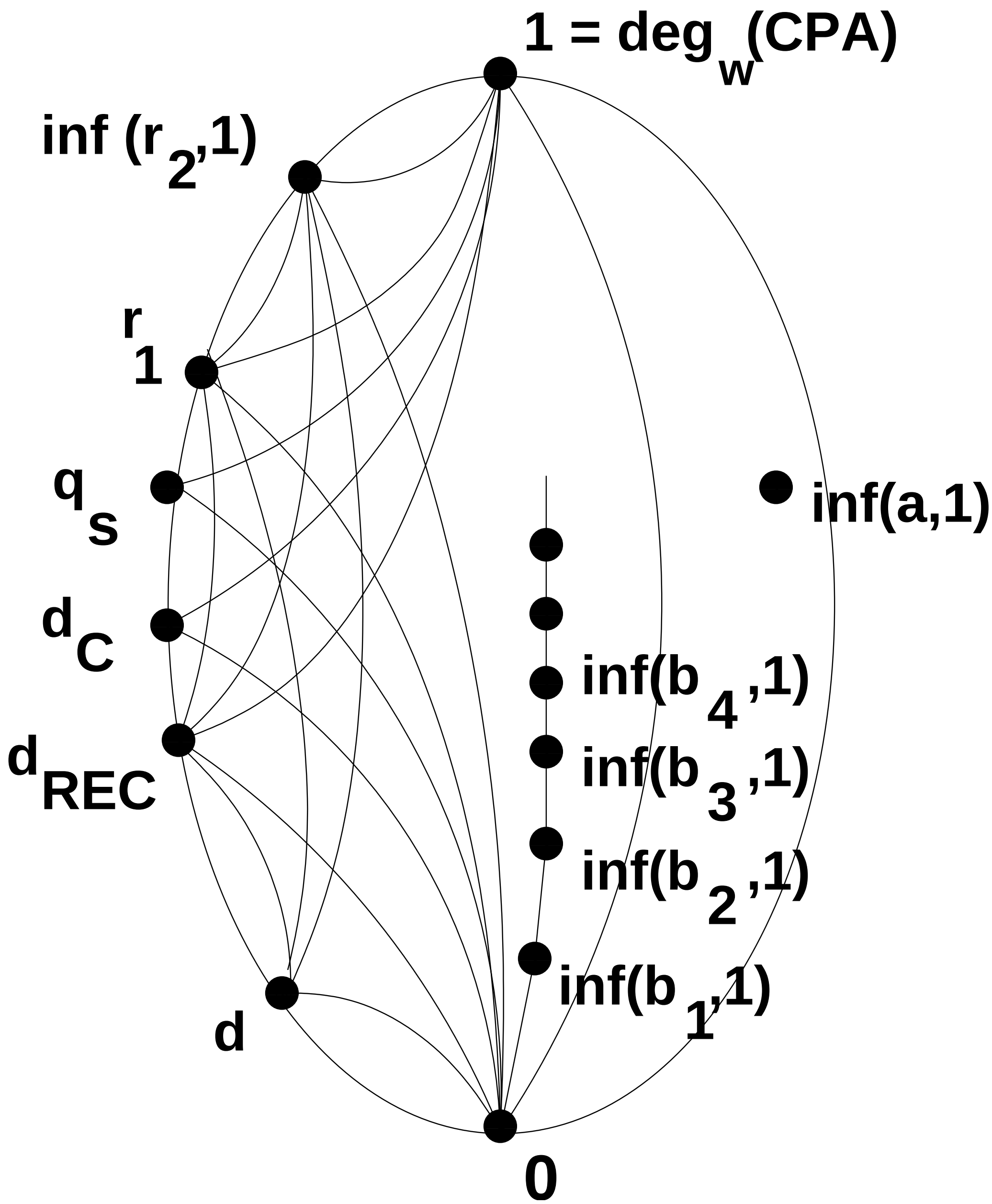
The proofs of these theorems concerning 2-dimensional subshifts of finite type are rather difficult. Cf. tilings of the plane.

Remark. Hochman and Meyerovitch have proved that a positive real number is the entropy of a 2-dimensional subshift of finite type if and only if it is the limit of a computable descending sequence of rational numbers.

Remark. An interesting research program is as follows. Given a 2-dimensional subshift of finite type, to correlate its dynamical properties with its degree of unsolvability.

Remark. My recent research on mass problems shows that there are many specific, natural degrees of unsolvability here. See also the next slide, where $\mathcal{E}_{\mathrm{w}}$ is the lattice of weak degrees of nonempty Π^0_1 subsets of $2^{\mathbb{N}}$.

Remark. Theorem 8 says that $\mathcal{E}_{\mathrm{w}}$ is the same as the lattice of weak degrees of 2-dimensional subshifts of finite type.



A picture of $\mathcal{E}_{\mathrm{w}}$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{r} =$ randomness, $\mathbf{b} =$ LR-reducibility, $\mathbf{q} =$ dimension, $\mathbf{d} =$ diagonal nonrecursiveness.

In classifying dynamical systems, the discovery of new invariants is extremely important. For instance, Kolmogorov introduced the entropy invariant in order to prove that the $(1/2, 1/2)$-Bernoulli shift and the $(1/3, 1/3, 1/3)$-Bernoulli shift are not measure-theoretically isomorphic.

Let X be a d-dimensional subshift of computable type. Then deg(X), the degree of unsolvability of the problem $P(X)$, is a topological invariant of X which appears to be new and different.

Compare deg(X) with ent(X), the topological entropy of X. Both deg(X) and ent(X) represent bounds on the complexity of the orbits of X, but these bounds are quite different. Namely, ent(X) is an upper bound (cf. the work of Lutz/Hitchcock/Mayordomo on effective Hausdorff dimension), while deg(X) is a lower bound.

Degrees of unsolvability.

Following Simpson 1999, let $\mathcal{E}_{\mathsf{w}}$ be the lattice of degrees of unsolvability associated with nonempty, effectively closed sets in $\{0,1\}^\infty$.

Many interesting degrees in $\mathcal{E}_{\mathsf{w}}$ are related to Kolmogorov complexity. For instance:

$\mathbf{d} = \deg(\{f \mid f \text{ is DNR}\})$.

$\mathbf{d}_C = \deg(\{f \mid f \text{ is DNR and } C\text{-bounded}\})$.

$\mathbf{d}_{\mathsf{REC}} = \deg(\{X \mid X \text{ is complex}\})$.

$\mathbf{q}_s = \deg(\{X \mid \dim(X) > s\})$.

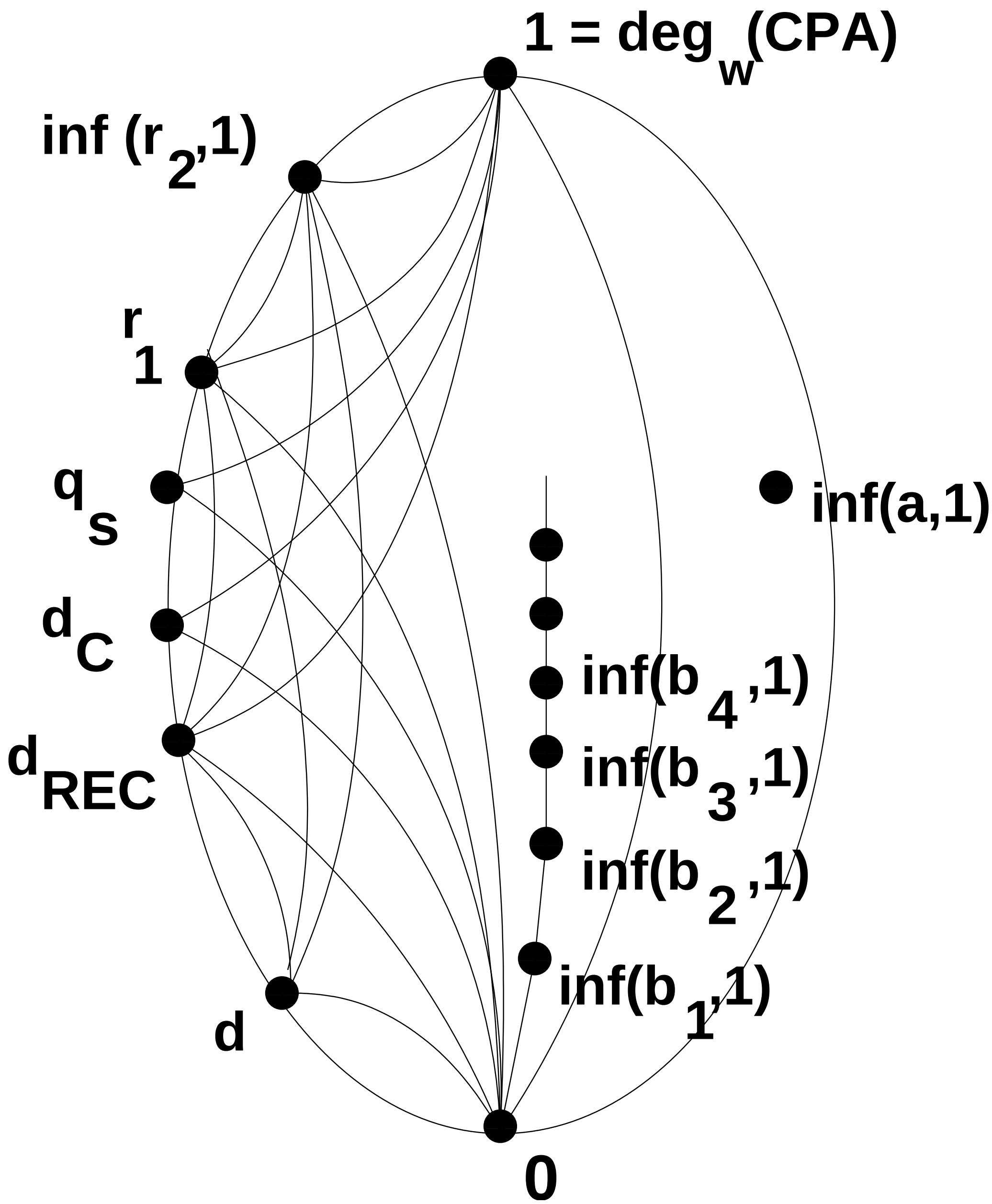
$\mathbf{b}_1 = \deg(\{X \mid 0' \leq_{\mathsf{LR}} X\})$ where $0'$ is the halting problem for Turing machines.

$\mathbf{b}_\alpha = \deg(\{X \mid 0^{(\alpha)} \leq_{\mathsf{LR}} X\})$ where $0^{(\alpha)}$ is the αth iterate of the Turing jump operator.

$\mathbf{r}_1 = \deg(\{X \mid X \text{ is random}\})$.

$\mathbf{r}_2 = \deg(\{X \mid X \text{ is random relative to } 0'\})$.

$\mathbf{1} = \deg(\{f \mid f \text{ is DNR and 2-bounded}\})$
$\quad = \deg(\{T \mid T \text{ is a completion of PA}\})$.

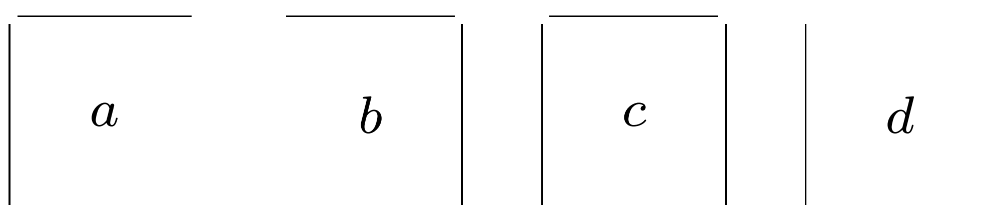


A picture of $\mathcal{E}_{\mathrm{w}}$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{r} =$ randomness, $\mathbf{b} =$ LR-reducibility, $\mathbf{q} =$ dimension, $\mathbf{d} =$ diagonal nonrecursiveness.

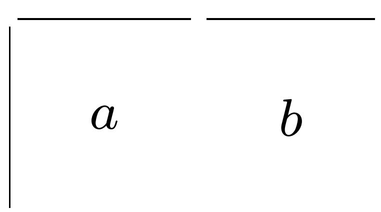
Tiling problems (Wang, 1961).

Let F be a finite set of *tiles*, i.e., 1×1 squares with colored edges. Let P_F be the *tiling problem* associated with F, i.e, the problem of covering the Euclidean plane with disjoint copies of tiles from F in such a way that adjacent edges have matching colors.

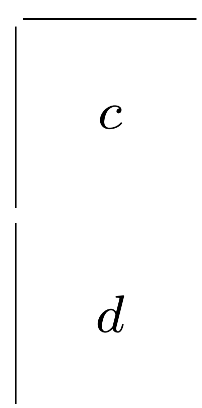
Example. Let F be this set of four tiles:



Then P_F is the problem of covering the plane with 2×1 and 1×2 rectangles

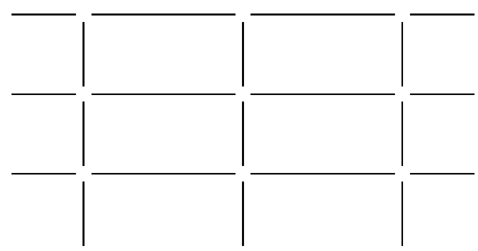


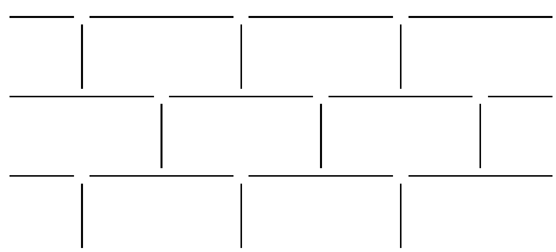
and

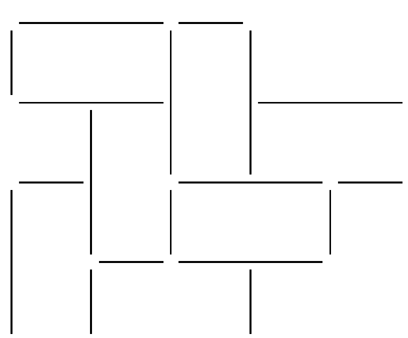


Example (continued).

The tiling problem P_F has many solutions:







Here are some theorems showing that tiling problems can be very difficult to solve.

Theorem (Berger, 1966)**.** We can construct a tiling problem P_F which has solutions but no periodic solution.

Theorem (Myers, 1974)**.** We can construct a tiling problem P_F which has solutions but no computable solution.

Theorem (Durand/Romashchenko/Shen, 2008)**.** We can construct a tiling problem with the following property. P_F has solutions, but $\mathsf{K}(S) \geq O(n) - O(1)$ for any $n \times n$ square S in any solution of P_F.

Note: $O(n) - O(1)$ is best possible.

Here is another theorem in this vein.

Theorem (Simpson, 2007)**.**
Let P_F be a tiling problem which has solutions. Then, the degree of unsolvability of P_F belongs to $\mathcal{E}_\mathsf{w}$. Conversely, each degree in $\mathcal{E}_\mathsf{w}$ is the degree of a tiling problem.

My paper proving this result has been accepted for publication in the journal *Ergodic Theory and Dynamical Systems*.

Remark. The study of tiling problems is essentially the same as 2-dimensional symbolic dynamics. Given a tiling problem P_F, the solution set S_F is either empty or a 2-dimensional shift space of finite type. Conversely, each 2-dimensional shift space of finite type is equivalent to the solution set of a tiling problem.

Current research.

One of my research projects is to study the relationship between the degree of unsolvability of P_F and the classical dynamical properties of the dynamical system S_F.

An classically important invariant of dynamical systems is entropy.

A less well-studied invariant is degree of unsolvability.

Both of these invariants measure the complexity of orbits in a dynamical system. The entropy is an upper bound, while the degree of unsolvability is a lower bound.

My pending NSF research proposal is entitled *Mass Problems and Symbolic Dynamics*.

Related activities.

I have given several talks on these ideas to dynamical systems audiences at Penn State and the University of Maryland.

In January 2009 I organized a session on *Logic and Dynamical Systems* at the Joint Mathematics Meetings in Washington, DC.

In February 2010 I will participate in a workshop on *Dynamics and Computation* at CIRM in Marseille, France. As part of the workshop I will lead a session on degrees of unsolvability and symbolic dynamics.

THE END.

THANK YOU!