# Computable Symbolic Dynamics

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Let $A$ be an *alphabet*, i.e., a finite set of symbols. Let $G$ be a computable group. We write $A^G = \{x \mid x : G \to A\}$.

Let $\sigma, \tau, \ldots$ range over functions $\sigma : F \to A$ where $F = \mathrm{dom}(\sigma)$ is a finite subset of $G$. The set of such functions is denoted $A^G_*$. For $\sigma \in A^G_*$ let

$$N_\sigma = \{x \in A^G \mid x \upharpoonright \mathrm{dom}(\sigma) = \sigma\}.$$

The $N_\sigma$'s are a basis for the standard product topology on $A^G$. Thus $C \subseteq A^G$ is topologically closed if and only if $C = A^G \setminus \bigcup_{\sigma \in S} N\sigma$ for some $S \subseteq A^G_*$. If $S$ is computable, we say that $C$ is *effectively closed* or $\Pi^0_1$.

The action of $G$ on $A^G$ is given by $x^g(h) = x(gh)$ for all $x \in A^G$ and $g, h \in G$. A $G$-*subshift* is a set $X \subseteq A^G$ which is nonempty, topologically closed, and closed under the action of $G$, i.e., $x \in X$ implies $x^g \in X$.

We write

$$\mathrm{Seq}(x) = \{\sigma \in A_*^G \mid \exists g\,(x^g \restriction \mathrm{dom}(\sigma) = \sigma)\}.$$

This notation is inspired by the book *Ramsey Theory* by Graham, Rothschild and Spencer. Given $E \subseteq A_*^G$ let

$$X_E = \{x \in A^G \mid \mathrm{Seq}(x) \cap E = \emptyset\}$$

provided this set is nonempty. Clearly $X_E$ is a subshift. We say that the subshift $X_E$ is defined by a set of *excluded configurations*, $E$. If $E$ is finite, we say that $X_E$ is *of finite type*. If $E$ is computable, we say that $X_E$ is *of computable type*. It can be shown that a subshift is of computable type if and only if it is effectively closed.

It is well known that any subshift is defined by a set of excluded configurations. In other words, given a subshift $X$ we can find $E \subseteq A_*^G$ such that $X = X_E$. It can be shown that most or all subshifts which arise in practice are of computable type. Here is a precise general result.

**Theorem 1.** Let $G$ act effectively on an effectively closed, effectively totally bounded set $Y$ in an effectively presented complete separable metric space. For each $a \in A$ let $C_a$ be an effectively closed subset of $Y$. Let

$$E = \{\sigma \mid \neg(\exists y \in Y)\,(\forall g \in \mathrm{dom}(\sigma))\,(y^g \in C_{x(g)})\}.$$

If $X_E \neq \emptyset$, then $X_E$ is a subshift of computable type.

Let $P(X)$ be the problem of finding a point of $X$. We have the following theorem.

**Theorem 2** (Michael Hochman, 2008). If $X$ is of computable type and *minimal* (i.e., every orbit is dense), then $P(X)$ is algorithmically solvable.

*Proof.* Write $\mathsf{Seq}(X) = \bigcup_{x \in X} \mathsf{Seq}(x)$. Because $X$ is minimal, we have

$$(\forall x, y \in X)\, \forall F\, \exists g\, (x^g \restriction F = y \restriction F),$$

i.e.,

$$(\forall x, y \in X)\, \forall F\, \forall g\, \exists h\, (x^g \restriction F = y^h \restriction F),$$

i.e.,

$$(\forall x, y \in X)\, (\mathsf{Seq}(x) = \mathsf{Seq}(y)).$$

Thus

$$\sigma \in \mathsf{Seq}(X) \Leftrightarrow (\forall x \in X)\, (\sigma \in \mathsf{Seq}(x))$$

$$\Leftrightarrow \forall x\, (x \in X \Rightarrow \exists g\, (x^g \restriction \mathrm{dom}(\sigma) = \sigma))$$

and a Tarski/Kuratowski computation shows that $\mathsf{Seq}(X)$ is $\Sigma^0_1$, i.e., it is the range of a computable sequence. On the other hand,

$$\begin{aligned}\sigma \in \mathsf{Seq}(X) &\Leftrightarrow (\exists x \in X)\, \exists g\, (x^g \restriction \mathsf{dom}(\sigma) = \sigma) \\ &\Leftrightarrow \exists y\, (y \in X \wedge y \restriction \mathsf{dom}(\sigma) = \sigma)\end{aligned}$$

and a Tarski/Kuratowski computation shows that $\mathsf{Seq}(X)$ is $\Pi^0_1$, i.e., it is the complement of a $\Sigma^0_1$ set. It follows that $\mathsf{Seq}(X)$ is $\Delta^0_1$, i.e., computable. Now fix a computable sequence of finite sets $\emptyset = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$ with $\bigcup_{n=0}^{\infty} F_n = G$. Starting with $\sigma_0 = \emptyset$ and given $\sigma_n \in \mathsf{Seq}(X)$ with $\mathsf{dom}(\sigma_n) = F_n$, search for $\sigma_{n+1} \in \mathsf{Seq}(X)$ extending $\sigma_n$ with $\mathsf{dom}(\sigma_{n+1}) = F_{n+1}$. Finally $x = \bigcup_{n=0}^{\infty} \sigma_n$ is a point of $X$ and is computable, Q.E.D.

**Remark.** Theorems 1 and 2 hold more generally, when $G$ is a recursively presented semigroup with identity.

If $G = \mathbb{Z}^d$ we say that $X$ is $d$-dimensional.

We now consider 1-dimensional subshifts, i.e., $G = \mathbb{Z}$.

**Theorem 3** ("classical"). Every 1-dimensional subshift of finite type contains periodic points.

**Corollary.** If $X$ is 1-dimensional and of finite type, then $P(X)$ is algorithmically solvable.

**Theorem 4** (Cenzer/Dashti/King, 2006). We can construct a 1-dimensional subshift $X$ of computable type such that $P(X)$ is algorithmically unsolvable.

**Theorem 5** (Joseph S. Miller, 2008). If $X$ is 1-dimensional of computable type, then $P(X)$ can have any desired degree of unsolvability.

This immediately implies the theorem of Cenzer/Dashti/King.

Restatement of Miller's theorem:

Given a nonempty effectively closed set $C \subseteq \{0,1\}^{\mathbb{N}}$, we can find a 1-dimensional subshift $X \subseteq \{0,1\}^{\mathbb{Z}}$ of computable type, plus computable functionals $\Phi : C \to X$ and $\Psi : X \to C$.

*Proof of Miller's theorem.* We write $\{0,1\}^* = \bigcup_{n=0}^{\infty}\{0,1\}^n = \{$finite strings of 0's and 1's$\}$. For $s \in \{0,1\}^*$ define $a_s, b_s \in \{0,1\}^*$ by induction on the length of $s$ as follows. Start with $a_\emptyset = 0$ and $b_\emptyset = 1$. Given $a_s$ and $b_s$ define

$$a_{s0} = a_s a_s a_s a_s b_s, \quad a_{s1} = a_s a_s a_s b_s b_s,$$

$$b_{s0} = a_s a_s b_s b_s b_s, \quad b_{s1} = a_s b_s b_s b_s b_s$$

and note that $a_s$ is the middle fifth of $a_{s0}$ and $a_{s1}$ while $b_s$ is the middle fifth of $b_{s0}$ and $b_{s1}$.

Given $C \subseteq \{0,1\}^{\mathbb{N}}$ let $Q_C = \bigcup_{y \in C} Q_y \subseteq \{0,1\}^{\mathbb{Z}}$ where

$$Q_y = \{x \mid \forall n\,(x \text{ is made of } a_{y\upharpoonright n}\text{'s and } b_{y\upharpoonright n}\text{'s})\}.$$

It is straightforward to show that if $C$ is nonempty and effectively closed then $Q_C$ is a subshift of computable type. Moreover, we have computable functionals $\Phi : C \to Q_C$ and $\Psi : Q_C \to C$ given by $\Phi(y) = \bigcup_{n=0}^{\infty} a_{y\upharpoonright n}$ and $\Psi(x) =$ the unique $y \in C$ such that $x \in Q_y$. Thus, letting $X = Q_C$, we have the desired result, Q.E.D.

**Remark.** For each $y \in \{0,1\}^{\mathbb{N}}$, $Q_y$ is a minimal subshift. Thus $Q_C$ is a dynamical system with the property that the orbit closure of every point is a minimal dynamical system. This property of dynamical systems is apparently somewhat unusual.

Now for the 2-dimensional case, $G = \mathbb{Z} \times \mathbb{Z}$.

**Theorem 6** (Berger, 1965). We can construct a 2-dimensional subshift of finite type which has no periodic points.

**Theorem 7** (Myers, 1974). We can construct a 2-dimensional subshift $X$ of finite type such that $P(X)$ is algorithmically unsolvable.

**Theorem 8** (Simpson, 2007). If $X$ is 2-dimensional of finite type, then $P(X)$ can have any desired degree of unsolvability.

In other words, given a nonempty effectively closed set $C \subseteq \{0,1\}^{\mathbb{N}}$, we can find a 2-dimensional subshift $X \subseteq \{0,1\}^{\mathbb{Z}\times\mathbb{Z}}$ of finite type along with computable functionals $\Phi : C \to X$ and $\Psi : X \to C$.

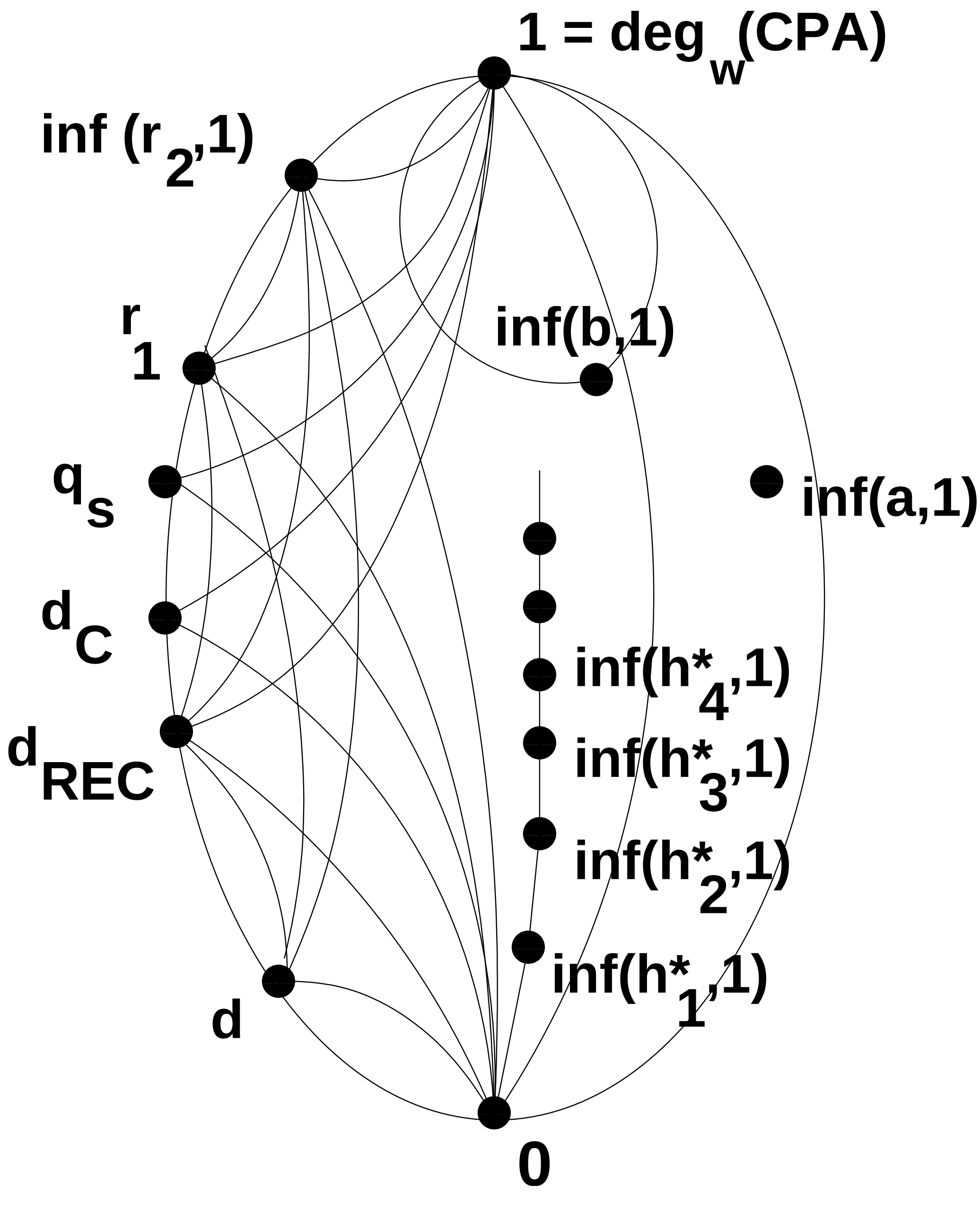
This immediately implies Myers's theorem, which immediately implies Berger's theorem.

The proofs of these theorems concerning 2-dimensional subshifts of finite type are rather difficult.

**Remark.** Hochman and Meyerovitch have proved that a positive real number is the entropy of a 2-dimensional subshift of finite type if and only if it is the limit of a computable descending sequence of rational numbers.

**Remark.** An interesting research program is as follows. Given a 2-dimensional subshift of finite type, to correlate its dynamical properties with its degree of unsolvability.

My recent research on mass problems shows that there are many specific, natural degrees of unsolvability here. See also the next slide, where $\mathcal{P}_w$ is the lattice of weak degrees of nonempty $\Pi^0_1$ subsets of $2^{\mathbb{N}}$. Theorem 8 says that $\mathcal{P}_w$ is the same as the lattice of weak degrees of 2-dimensional subshifts of finite type.




A picture of $\mathcal{P}_w$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{h} =$ hyperarithmeticity, $\mathbf{r} =$ randomness, $\mathbf{b} =$ a.e. domination, $\mathbf{q} =$ dimension, $\mathbf{d} =$ diagonal nonrecursiveness.

In classifying dynamical systems, the discovery of new invariants is extremely important. For instance, Kolmogorov introduced the entropy invariant in order to prove that the $(1/2, 1/2)$-Bernoulli shift and the $(1/3, 1/3, 1/3)$-Bernoulli shift are not measure-theoretically isomorphic.

Let $X$ be a $d$-dimensional subshift of computable type. Then deg($X$), the degree of unsolvability of the problem $P(X)$, is a topological invariant of $X$ which appears to be new and different.

Compare deg($X$) with ent($X$), the topological entropy of $X$. Both deg($X$) and ent($X$) represent bounds on the complexity of the orbits of $X$, but these bounds are quite different. Namely, ent($X$) is an upper bound (cf. the work of Lutz/Hitchcock/Mayordomo on effective Hausdorff dimension), while deg($X$) is a lower bound.