MASS PROBLEMS

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Abstract:

Informally, mass problems are similar to decision problems. The difference is that, while a decision problem has only one solution, a mass problem is allowed to have more than one solution. Many concepts which apply to decision problems apply equally well to mass problems. For instance, a mass problem is said to be solvable if it has at least one computable solution. Also, one mass problem is said to be reducible to another mass problem if, given a solution of the second problem, we can use it to find a solution of the first problem.

Many unsolvable mathematical problems are most naturally viewed as mass problems rather than decision problems. For example, let CPA be the problem of finding a completion of Peano Arithmetic. A well-known theorem going back to Gödel and Tarski says that CPA is unsolvable, in the sense that there are no computable completions of Peano Arithmetic. In describing CPA as a "problem" whose "solutions" are the completions of Peano Arithmetic, we are implicitly viewing CPA as a mass problem rather than a decision problem. This is because Peano Arithmetic has many different completions, with many different Turing degrees. There is no single Turing degree which can be said to measure the amount of unsolvability which is inherent in CPA.

Formally, a *mass problem* is defined to be an arbitrary set of Turing oracles, i.e., an arbitrary subset of the *Baire space*, ω^ω. Let P and Q be mass problems. We say that P is *solvable* if P has at least one recursive element. We say that P is *weakly reducible* to Q if for all $g \in Q$ there exists $f \in P$ such that f is Turing reducible to g. A *weak degree* is an equivalence class of mass problems under mutual weak reducibility. The weak degrees are partially ordered in the obvious way, by weak reducibility. Under this partial ordering, it is straightforward to show that the weak degrees form a complete distributive lattice. There is an obvious, natural embedding of the upper semilattice of Turing degrees into the

lattice of weak degrees, obtained by mapping the Turing degree of f to the weak degree of the singleton set $\{f\}$.

The lattice of weak degrees was first introduced by Albert Muchnik (of Friedberg/Muchnik fame) in 1963. Subsequently, weak degrees have been used to classify unsolvable problems which arise in various areas of mathematics including symbolic dynamics, algorithmic randomness, and model theory.

Let $\mathcal{D}_w$ be the lattice of all weak degrees. Let $\mathcal{P}_w$ be the sublattice of $\mathcal{D}_w$ consisting of the weak degrees of mass problems associated with nonempty Π^0_1 subsets of the *Cantor space*, $\{0,1\}^\omega$. Recent research beginning in 1999 has revealed that $\mathcal{P}_w$ is mathematically rich and contains many specific, natural, weak degrees corresponding to specific, natural, mass problems which are of great interest. For example, the top degree in $\mathcal{P}_w$ is the weak degree of CPA, the set of completions of Peano Arithmetic. Other specific, natural, weak degrees in $\mathcal{P}_w$ are closely related to various foundationally interesting topics: algorithmic randomness, reverse mathematics, hyperarithmeticity, diagonal nonrecursiveness, effective Hausdorff dimension, almost everywhere domination, subrecursive hierarchies, resource-bounded computational complexity, Kolmogorov complexity.

In this talk we introduce $\mathcal{P}_w$ and to survey some recent discoveries regarding specific, natural, weak degrees in $\mathcal{P}_w$.

Motivation:

Let $\mathcal{D}_T$ be the upper semilattice of all Turing degrees, a.k.a., "degrees of unsolvability."

In $\mathcal{D}_T$ there are a great many specific, interesting Turing degrees, namely

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \cdots < \mathbf{0}^{(\alpha)} < \mathbf{0}^{(\alpha+1)} < \cdots$$

where α runs through (a large initial segment of) the countable ordinal numbers (depending on whether V=L or not . . .). See my paper *The hierarchy based on the jump operator*, Kleene Symposium, North-Holland, 1980.

Historically, the original purpose of $\mathcal{D}_T$ (Turing 1936, Kleene/Post 1940's, 1950's) was to serve as a framework for classifying unsolvable mathematical problems.

In the 1950's, 1960's, and 1970's, it turned out that many specific, natural, well-known, unsolvable mathematical problems are indeed of Turing degree $\mathbf{0}'$:

- the Halting Problem for Turing machines (Turing's original example)

- the Word Problem for finitely presented groups

- the Triviality Problem for finitely presented groups, etc.

- Hilbert's 10th Problem for Diophantine equations

- and many others.

In addition, the **arithmetical hierarchy**

$$\mathbf{0}^{(n)}, \quad n < \omega$$

and the **hyperarithmetical hierarchy**

$$\mathbf{0}^{(\alpha)}, \quad \alpha < \omega_1^{\mathsf{CK}}$$

have been useful in studying the foundations of mathematics.

These hierarchies, based on iterating the Turing jump operator, have been useful precisely because of their ability to classify unsolvable mathematical problems.

This aspect of $\mathcal{D}_T$ is explored in my book *Subsystems of Second Order Arithmetic*, Springer-Verlag, 1999, which is the basic reference on reverse mathematics.

On the other hand, there are many unsolvable mathematical problems which do not fit into the $\mathcal{D}_T$ framework at all.

For example, consider the following problem, which we call CPA:

> To find a complete, consistent theory which includes Peano Arithmetic.

Note that CPA is a very natural problem, in view of the Gödel Incompleteness Theorem, which says that Peano Arithmetic itself is incomplete.

Moreover, by the work of Tarski, Gödel, and Rosser, the problem CPA is "unsolvable" in the sense that there is no *computable*, complete, consistent theory which includes Peano Arithmetic.

However (and this is the interesting point), it is not possible to assign a specific Turing degree ("degree of unsolvability") to the unsolvable problem CPA.

CPA is this unsolvable problem:

> To find a complete, consistent theory which includes Peano Arithmetic.

Although CPA is unsolvable, *there is no one specific Turing degree* associated to CPA. Thus, the Turing degree framework fails to classify CPA.

Digression: One may consider the Turing degree $\mathbf{0}^{(\omega)}$. It is reasonable to associate $\mathbf{0}^{(\omega)}$ to True Arithmetic, which is one particular, complete, consistent extension of Peano Arithmetic. However, it is unreasonable to associate $\mathbf{0}^{(\omega)}$ to the problem CPA as a whole. This is because, beyond True Arithmetic, there are many other complete, consistent extensions of Peano Arithmetic. Some of them even have Turing degree $< \mathbf{0}'$.

If we want to classify unsolvable problems such as CPA, we need a different framework.

The appropriate framework is:

MASS PROBLEMS.

Here are some more examples.

R_1: To find an infinite sequence of 0's and 1's which is *random* in the sense of Martin-Löf.

R_n: To find an infinite sequence of 0's and 1's which is n-*random*, $n = 1, 2, \ldots$.

DNR: To find a function f which is *diagonally nonrecursive*, i.e., $f(n) \neq \varphi_n^{(1)}(n)$ for all n.

$\mathrm{DNR}_{\mathrm{REC}}$: To find a function f which is diagonally nonrecursive and *recursively dominated*, i.e., there exists a recursive function g such that $f(n) < g(n)$ for all n.

AED: To find a Turing oracle A which is *almost everwhere dominating*, i.e., with probability 1, every function which is computable from a sequence of coin tosses is dominated by some function which is computable from A.

Each of the problems R_1, R_2, ..., DNR, $\mathrm{DNR}_{\mathrm{REC}}$, AED, ..., is similar to the problem CPA. In each case, the problem is *unsolvable* (i.e., there is no computable solution), but *there is no one specific Turing degree* that can be attached to the problem.

In other words, each of the problems

CPA, R_1, R_2, . . . , DNR, DNR_{REC}, AED, . . . ,

is an example of an *unsolvable mass problem*.

If we wish to classify unsolvable problems of this kind, we need a concept of "degree of unsolvability" which is more general than the Turing degrees.

The appropriately generalized concept of "degree of unsolvability" is:

WEAK DEGREES,

also known as

MUCHNIK DEGREES.

This concept of "degree of unsolvability" is the one that has turned out to be most useful for classification and comparison of specific, natural, unsolvable mass problems.

Mass problems (informal discussion):

A "decision problem" is the problem of deciding whether a given $n \in \omega$ belongs to a fixed set $A \subseteq \omega$ or not. To compare decision problems, we use Turing reducibility. $A \leq_T B$ means that A can be computed using an oracle for B.

A "mass problem" is a problem with a not necessarily unique solution. (By contrast, a "decision problem" has only one solution.)

The "mass problem" associated with a set $P \subseteq \omega^\omega$ is the "problem" of computing an element of P.

The "solutions" of P are the elements of P.

One mass problem is said to be "reducible" to another if, given any solution of the second problem, we can use it as an oracle to compute a solution of the first problem.

Rigorous definition:

Let P and Q be subsets of ω^ω.

We view P and Q as mass problems.

We say that P is *weakly reducible* to Q (i.e., P is *Muchnik reducible* to Q), if

$$(\forall Y \in Q)\ (\exists X \in P)\ (X \leq_T Y)\ .$$

This is abbreviated $P \leq_w Q$.

Summary:

$P \leq_w Q$ means that, given any solution of Q, we can use it as an oracle to compute a solution of P.

Digression: weak vs. strong reducibility

Let P and Q be subsets of ω^ω.

1. P is *weakly reducible* to Q, $P \leq_w Q$, if for all $Y \in Q$ there exists e such that $\{e\}^Y \in P$.

2. P is *strongly reducible* to Q, $P \leq_s Q$, if there exists e such that $\{e\}^Y \in P$ for all $Y \in Q$.

Strong reducibility is a uniform variant of weak reducibility. By a result of Nerode, there is an analogy:

$$\frac{\text{weak reducibility}}{\text{Turing reducibility}} = \frac{\text{strong reducibility}}{\text{truth table reducibility}}.$$

In this talk we deal only with weak reducibility.

Historical note:

Weak reducibility is due to Muchnik 1963.

Strong reducibility is due to Medvedev 1955.

The lattice $\mathcal{P}_w$:

We focus on Π^0_1 subsets of 2^ω, i.e., $P = \{\text{paths through } T\}$ where T is a recursive subtree of $2^{<\omega}$, the full binary tree of finite sequences of 0's and 1's. Two of the earliest pioneering papers on Π^0_1 subsets of 2^ω are by Jockusch/Soare 1972.

We define $\mathcal{P}_w$ to be the set of weak degrees of nonempty Π^0_1 subsets of 2^ω, ordered by weak reducibility.

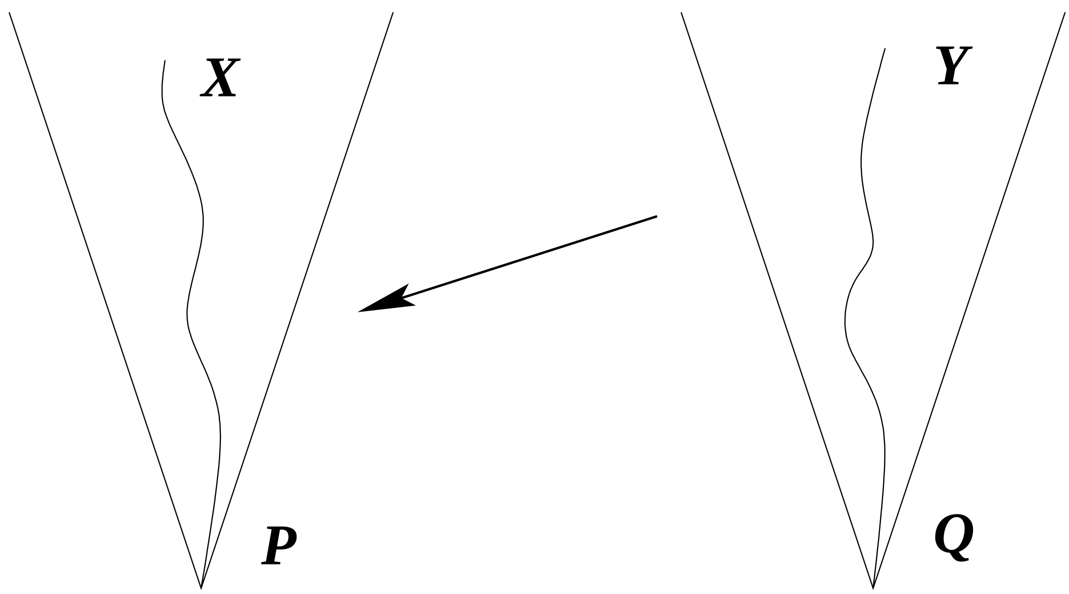
Basic facts about $\mathcal{P}_w$:

1. $\mathcal{P}_w$ is a distributive lattice, with sup given by $P \times Q = \{X \oplus Y \mid X \in P, Y \in Q\}$, and inf given by $P \cup Q$.

2. The bottom element of $\mathcal{P}_w$ is the weak degree of 2^ω.

3. The top element of $\mathcal{P}_w$ is the weak degree of CPA = {completions of Peano Arithmetic}. (see Scott/Tennenbaum, Jockusch/Soare).

weak reducibility of Π^0_1 subsets of 2^ω:



$P \leq_w Q$ means:
$(\forall Y \in Q)\ (\exists X \in P)\ (X \leq_T Y)$.

P, Q are given by infinite recursive subtrees of the full binary tree of finite sequences of 0's and 1's.

X, Y are infinite (nonrecursive) paths through P, Q respectively.

The lattice $\mathcal{P}_w$ (review):

A *weak degree* is an equivalence class of subsets of ω^ω under the equivalence relation $P \leq_w Q$ and $Q \leq_w P$. The weak degrees have a partial ordering induced by $\leq_w$.

We define $\mathcal{P}_w$ to be the set of weak degrees of nonempty Π^0_1 subsets of 2^ω, partially ordered by weak reducibility.

$\mathcal{P}_w$ is a countable distributive lattice.

The bottom element of $\mathcal{P}_w$ is the weak degree of 2^ω.

The top element of $\mathcal{P}_w$ is the weak degree of

$$\text{CPA} = \{\text{completions of Peano Arithmetic}\}.$$

We use $\mathbf{1}$ to denote this weak degree.

Embedding $\mathcal{E}_T$ into $\mathcal{P}_w$:

Theorem (Simpson 2002)**:**
There is a natural embedding $\phi : \mathcal{E}_T \to \mathcal{P}_w$.

($\mathcal{E}_T =$ the semilattice of Turing degrees of r.e. subsets of ω. $\mathcal{P}_w =$ the lattice of weak degrees of nonempty Π^0_1 subsets of 2^ω.)

The embedding ϕ is given by

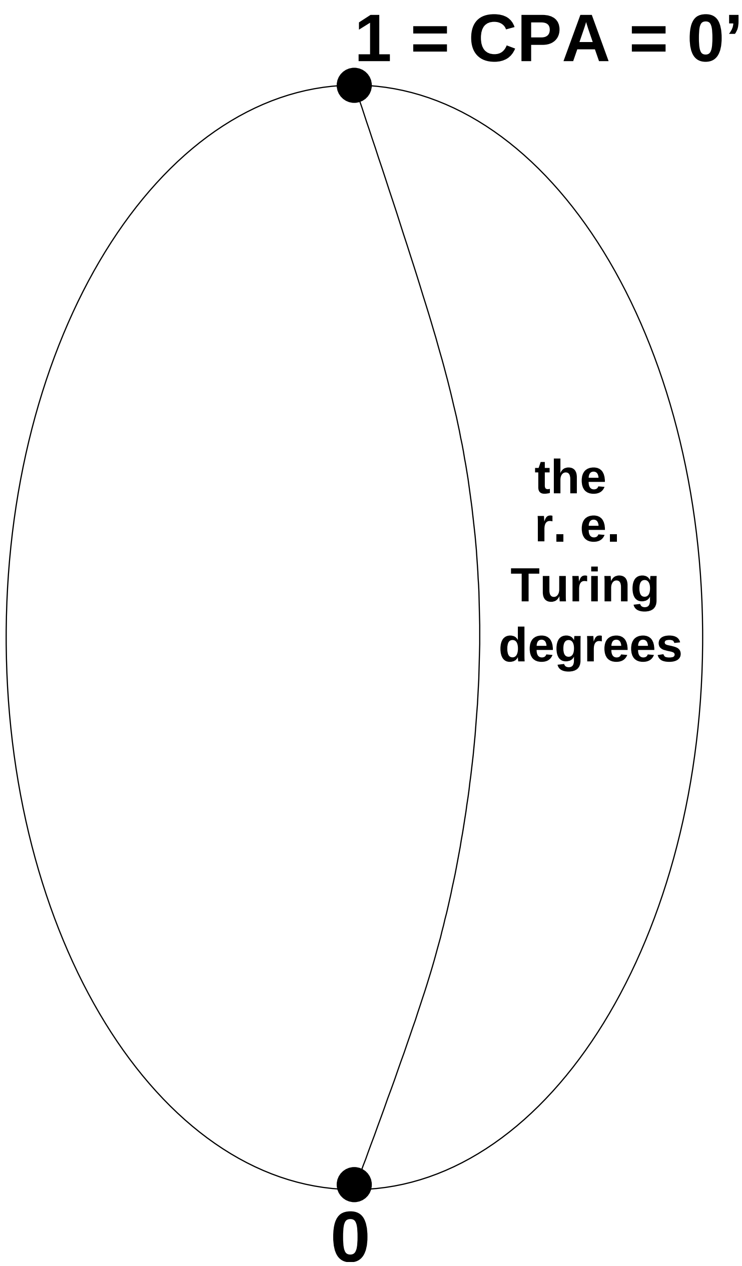
$$\phi : \text{deg}_T(A) \mapsto \text{deg}_w(\text{CPA} \cup \{A\}).$$

Note that $\text{CPA} \cup \{A\}$ is not a Π^0_1 set. However, it is of the same weak degree as a Π^0_1 set. This is already a nontrivial result.

The embedding ϕ is one-to-one and preserves $\leq$, sup, and the top and bottom elements.

Convention: We sometimes identify $\mathcal{E}_T$ with its image in $\mathcal{P}_w$. In particular, we sometimes identify $\mathbf{0}', \mathbf{0} \in \mathcal{E}_T$ with $\mathbf{1}, \mathbf{0} \in \mathcal{P}_w$, the top and bottom elements of $\mathcal{P}_w$.

A picture of the lattice $\mathcal{P}_w$:



$\mathcal{E}_T$ is embedded in $\mathcal{P}_w$. $\mathbf{0}'$ and $\mathbf{0}$ are the top and bottom elements of both $\mathcal{E}_T$ and $\mathcal{P}_w$.

Structural properties of $\mathcal{P}_w$:

1. $\mathcal{P}_w$ is a countable distributive lattice. Every countable distributive lattice is lattice embeddable in every initial segment of $\mathcal{P}_w$. (Binns/Simpson 2001)

2. The $\mathcal{P}_w$ analog of the Sacks Splittting Theorem holds. (Binns, 2002)

3. We conjecture that the $\mathcal{P}_w$ analog of the Sacks Density Theorem holds.

These structural results for $\mathcal{P}_w$ are proved by means of priority arguments, just as for $\mathcal{E}_T$.

4. Within $\mathcal{P}_w$ the degrees $\mathbf{r}_1$ and $\inf(\mathbf{r}_2, \mathbf{1})$ are meet irreducible and do not join to $\mathbf{1}$. (Simpson 2002, 2004)

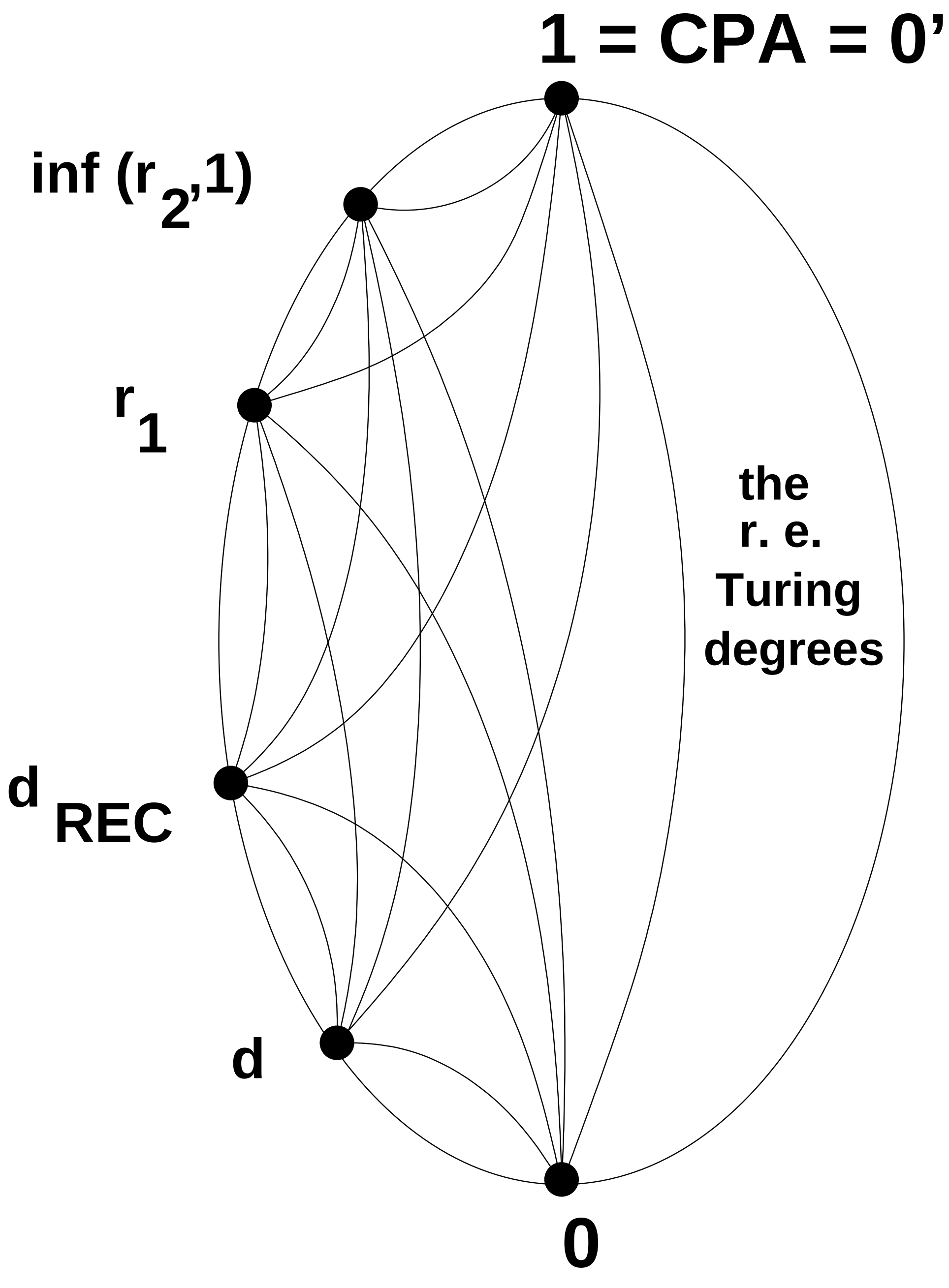
5. $\mathbf{0}$ is meet irreducible. (This is trivial.)

Natural examples in $\mathcal{P}_w$:

In the $\mathcal{P}_w$ context, we have discovered many specific, natural degrees which are $> \mathbf{0}$ and $< \mathbf{1}$.

The specific, natural degrees in $\mathcal{P}_w$ which we have discovered are related to foundationally interesting topics:

- algorithmic randomness,
- diagonal nonrecursiveness,
- reverse mathematics,
- subrecursive hierarchies,
- computational complexity (PTIME, EXPTIME, . . .),
- Kolmogorov complexity,
- effective Hausdorff dimension,
- hyperarithmeticity.



Note: Except for $\mathbf{0}'$ and $\mathbf{0}$, the r.e. Turing degrees are incomparable with these specific, natural degrees in $\mathcal{P}_w$.

Some specific, natural degrees in $\mathcal{P}_w$:

$\mathbf{r}_n =$ the weak degree of the set of n-random reals.

$\mathbf{d} =$ the weak degree of the set of diagonally nonrecursive functions.

$\mathbf{d}_{\mathsf{REC}} =$ the weak degree of the set of diagonally nonrecursive functions which are recursively dominated.

Theorem (Simpson 2002, Ambos $\cdots$ 2004)**:**
In $\mathcal{P}_w$ we have

$$\mathbf{0} < \mathbf{d} < \mathbf{d}_{\mathsf{REC}} < \mathbf{r}_1 < \inf(\mathbf{r}_2, \mathbf{1}) < \mathbf{1}.$$

Theorem (Simpson 2004)**:**

1. $\mathbf{r}_1$ is the maximum weak degree of a Π^0_1 subset of 2^ω which is of positive measure.

2. $\inf(\mathbf{r}_2, \mathbf{1})$ is the maximum weak degree of a Π^0_1 subset of 2^ω whose Turing upward closure is of positive measure.

Structural properties of $\mathcal{P}_w$:

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2. The $\mathcal{P}_w$ analog of the Sacks Splittting Theorem holds. (Stephen Binns, 2002)

3. We conjecture that the $\mathcal{P}_w$ analog of the Sacks Density Theorem holds.

These structural results for $\mathcal{P}_w$ are proved by means of priority arguments, just as for $\mathcal{E}_T$.

4. Within $\mathcal{P}_w$ the degrees $\mathbf{r}_1$ and $\inf(\mathbf{r}_2, \mathbf{1})$ are meet irreducible and do not join to $\mathbf{1}$. (Simpson 2002, 2004)

5. $\mathbf{0}$ is meet irreducible. (This is trivial.)

Another source of specific degrees in $\mathcal{P}_w$:

almost everywhere domination.

Definition (Dobrinen/Simpson 2003)**:**

B is *almost everywhere dominating* if,
for almost all $X \in 2^\omega$, each function $\leq_T X$
is dominated by some function $\leq_T B$.

Here "almost all" refers to
the fair coin measure on 2^ω.

Randomness and a.e. domination
are closely related to
the reverse mathematics of measure theory.
See Brown/Giusto/Simpson 2002,
Dobrinen/Simpson 2004, and
Kjos-Hanssen/Miller/Solomon 2008.

Some additional, natural degrees in $\mathcal{P}_w$:

Let $\mathbf{b}_1 = \text{deg}_w(\text{AED})$ where

$$\text{AED} = \{B \mid B \text{ is a. e. dominating}\}.$$

Let $\mathbf{b}_2 = \text{deg}_w(\text{AED} \times \text{R}_1)$ where

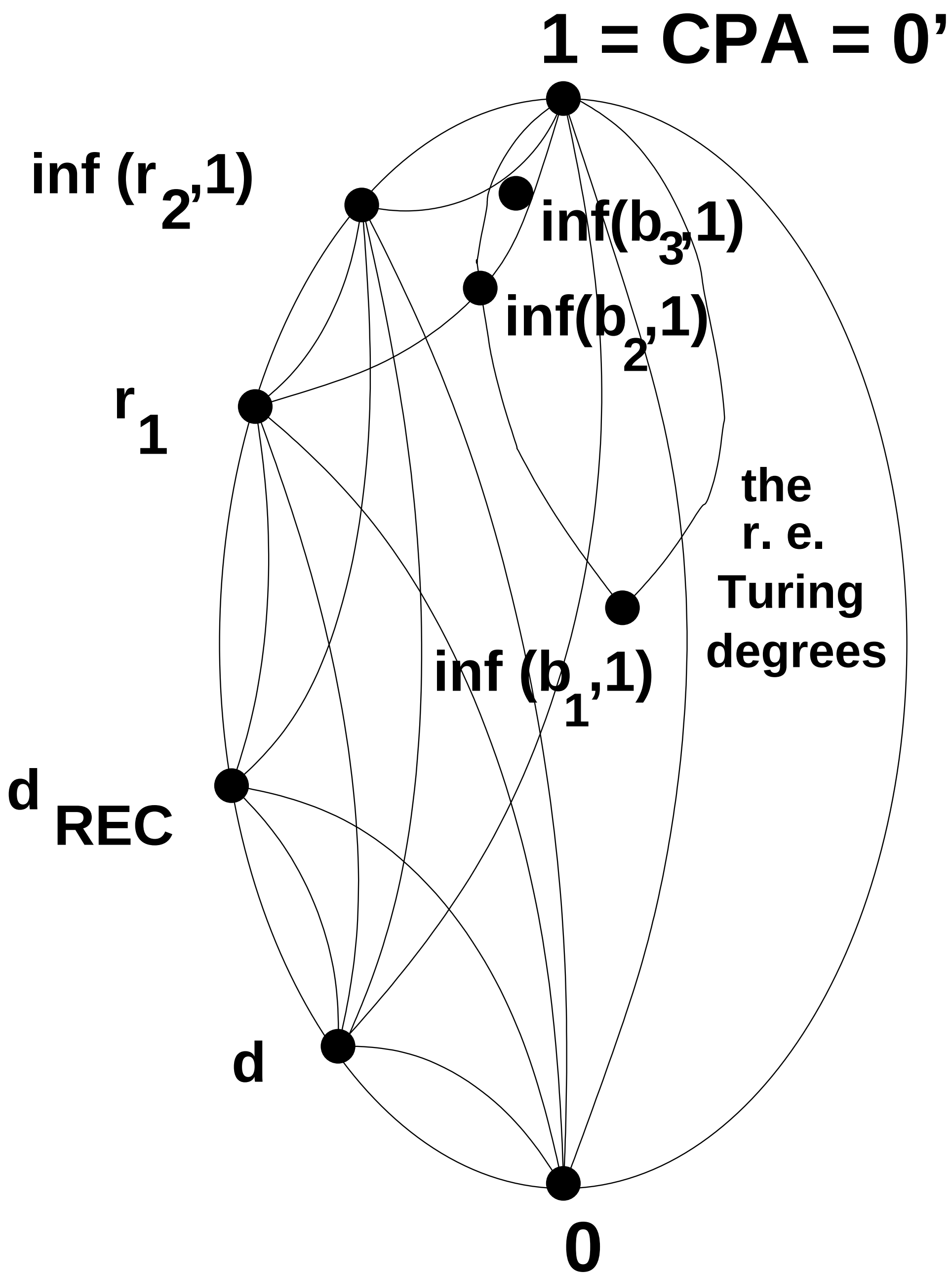
$$\text{R}_1 = \{A \mid A \text{ is 1-random}\}.$$

Let $\mathbf{b}_3 = \text{deg}_w(\text{AED} \cap \text{R}_1)$.

Theorem (Simpson, 2006)**:** In $\mathcal{P}_w$ we have:

- $\mathbf{0} < \inf(\mathbf{b}_1, \mathbf{1}) < \inf(\mathbf{b}_2, \mathbf{1}) < \inf(\mathbf{b}_3, \mathbf{1}) < \mathbf{1}$.
- $\inf(\mathbf{b}_1, \mathbf{1}) <$ some r.e. degrees $< \mathbf{0}'$.
- $\inf(\mathbf{b}_2, \mathbf{1}) \mid$ all r.e. degrees except $\mathbf{0}, \mathbf{0}'$.
- $\inf(\mathbf{b}_3, \mathbf{1}) >$ some r.e. degrees $> \mathbf{0}$.

The proof uses many interesting known results concerning randomness and almost everywhere domination (Cholak, Greenberg, Miller, Binns, Kjos-Hanssen, Lerman, Solomon, Hirschfeldt, Nies, Stephan, . . .).



Note that $\inf(\mathbf{b}_1, \mathbf{1})$ and $\inf(\mathbf{b}_3, \mathbf{1})$, unlike $\inf(\mathbf{b}_2, \mathbf{1})$, are comparable with some r.e. Turing degrees other than $\mathbf{0}'$ and $\mathbf{0}$.

Some additional, specific degrees in $\mathcal{P}_w$:

Definition: $\mathbf{d}_{\mathsf{REC}}$ = the weak degree of the set of recursively dominated DNR functions.

Theorem (Kjos-Hanssen/Merkle/Stephan)**:** $\mathbf{d}_{\mathsf{REC}}$ = the weak degree of

$\{A \in 2^\omega \mid (\exists f \in \mathsf{REC})\, \forall n\, (K(A \upharpoonright n) > f^{-1}(n))\}$.

Here K denotes Kolmogorov complexity.

Definition (Simpson 2004)**:** $\mathbf{d}_\alpha$ = the weak degree of the set of DNR functions dominated by some $f \in \mathsf{REC}_\alpha$. Here REC_α is the Wainer hierarchy, $\alpha \leq \varepsilon_0$.

Theorem (Kjos-Hanssen/Simpson 2006)**:** $\mathbf{d}_\alpha$ = the weak degree of

$\{A \mid (\exists f \in \mathsf{REC}_\alpha)\, \forall n\, (K(A \upharpoonright n) > f^{-1}(n))\}$.

Ambos-Spies/Kjos-Hanssen/Lempp/Slaman 2004 and Simpson 2005 have shown that in $\mathcal{P}_w$ we have

$$\mathbf{r}_1 > \mathbf{d}_0 > \mathbf{d}_1 > \cdots > \mathbf{d}_\alpha > \cdots > \mathbf{d}_{\mathsf{REC}}.$$

A remarkable refinement:

Definition (Lutz, Mayordomo)**:**
For each point $X \in 2^\omega$,
the *effective Hausdorff dimension of* X is

$$\text{edim}(X) \;=\; \liminf_{n\to\infty} \frac{K(X \upharpoonright n)}{n}$$

where K denotes Kolmogorov complexity.

Fact: For each Π^0_1 set $P \subseteq 2^\omega$,
the Hausdorff dimension of P is given by

$$\dim(P) \;=\; \sup\{\text{edim}(X) \mid X \in P\}\ .$$

Definition: Given a right recursively enumerable real number r in the range $0 \le r < 1$, let $\mathbf{q}_r$ be the weak degree of

$$Q_r \;=\; \{X \in 2^\omega \mid \text{edim}(X) > r\}\ .$$

It can be shown that $\mathbf{q}_r \in \mathcal{P}_w$.

Theorem (J. Miller)**:** For $0 \le r < s < 1$
we have $\mathbf{d}_0 < \mathbf{q}_r < \mathbf{q}_s < \mathbf{r}_1$.

Some additional examples in $\mathcal{P}_w$:

Definition:
$\mathbf{d}^2 =$ the weak degree of the set of $f \oplus g$ such that f is diagonally nonrecursive, and g is diagonally nonrecursive relative to f. More generally, define $\mathbf{d}^n$ for all $n \geq 1$. This can be extended into the transfinite.

Theorem (A-S/K-H/L/S, Simpson)**:**
In $\mathcal{P}_w$ we have

$$\mathbf{r}_1 > \mathbf{d}_0 > \mathbf{d}_1 > \cdots > \mathbf{d}_\alpha > \cdots > \mathbf{d}_{\mathsf{REC}}$$

and

$$\mathbf{d} = \mathbf{d}^1 < \mathbf{d}^2 < \cdots < \mathbf{d}^n < \cdots < \mathbf{r}_1 \,.$$

We conjecture that $\mathbf{d}^n$ is incomparable with $\mathbf{d}_\alpha$ and with $\mathbf{d}_{\mathsf{REC}}$. This would be another example of specific, natural degrees in $\mathcal{P}_w$ which are incomparable with each other.

Index sets in $\mathcal{P}_w$:

Here is a result indicating that $\mathcal{P}_w$ partakes of hyperarithmeticity.

Let P_i, $i \in \omega$ be the standard enumeration of all nonempty Π^0_1 subsets of 2^ω.
Let $\mathbf{p}_i = \deg_w(P_i)$.
By definition, $\mathcal{P}_w = \{\mathbf{p}_i \mid i \in \omega\}$.

Theorem (Cole/Simpson 2006)**:**
The index set $\{i \mid \mathbf{p}_i = \mathbf{1}\}$ is Π^1_1 complete.

More generally:

Theorem (Cole/Simpson 2006)**:**
For any j such that $\mathbf{p}_j > \mathbf{0}$, the index sets $\{i \mid \mathbf{p}_i = \mathbf{p}_j\}$ and $\{i \mid \mathbf{p}_i \geq \mathbf{p}_j\}$ are Π^1_1 complete.

Problem: Characterize the j's for which $\{i \mid \mathbf{p}_i \leq \mathbf{p}_j\}$ is Π^1_1 complete.

Embedding hyperarithmeticity into $\mathcal{P}_w$:

Definition (Cole/Simpson 2006):
A function $f(n)$ is said to be *boundedly limit recursive* in a Turing oracle A, abbreviated $f \in \mathsf{BLR}(A)$, if there exist an A-recursive approximating function $\widetilde{f}(n,s)$ and a recursive bounding function $\widehat{f}(n)$ such that for all n, $f(n) = \lim_s \widetilde{f}(n,s)$ and $|\{s \mid \widetilde{f}(n,s) \neq \widetilde{f}(n,s+1)\}| < \widehat{f}(n)$.

Definition (Cole/Simpson 2006):
For $\alpha < \omega_1^{\mathsf{CK}}$ let $\mathbf{h}^*_\alpha =$ the weak degree of

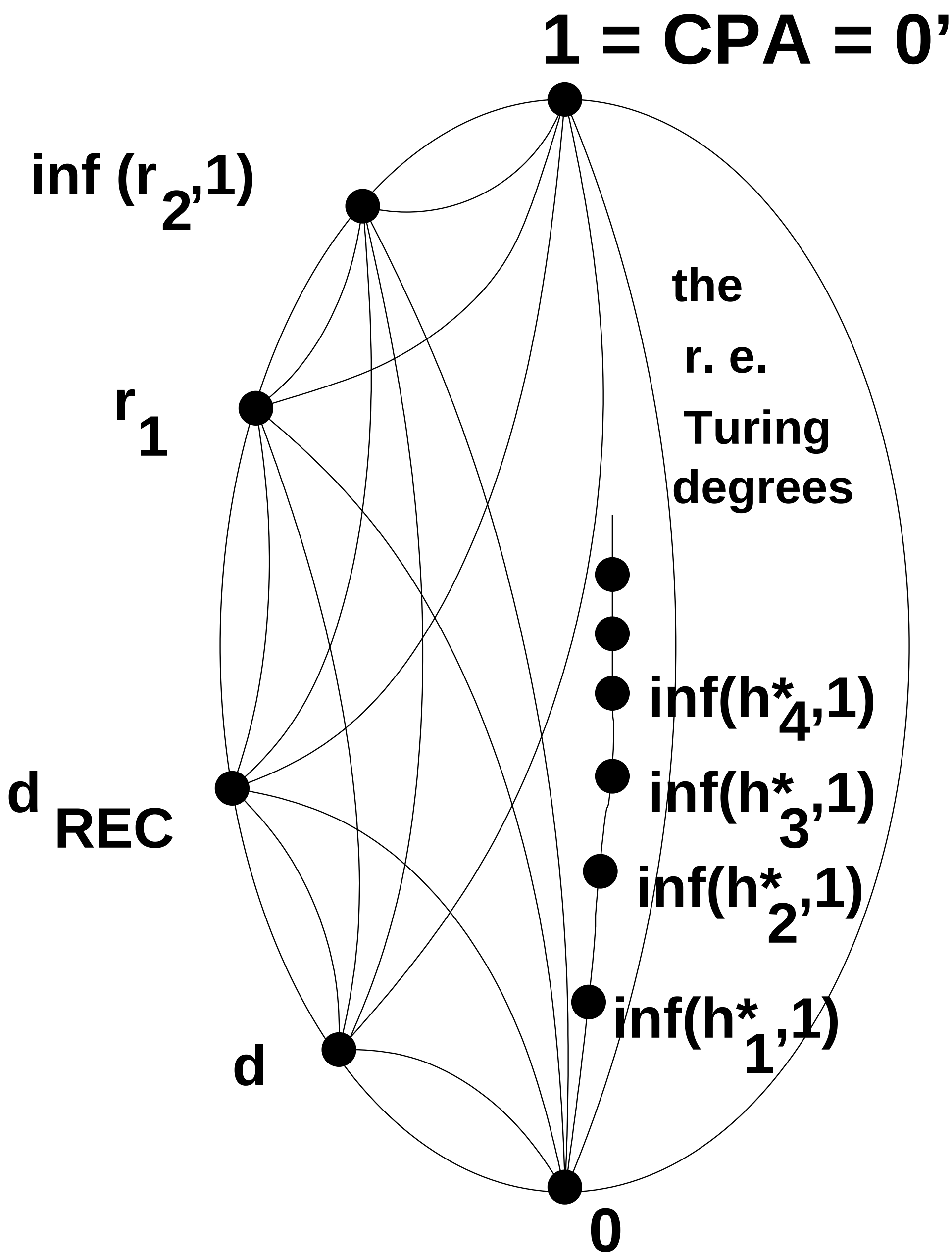
$$\left\{A \mid \mathsf{BLR}(0^{(\alpha)}) \subseteq \mathsf{BLR}(A)\right\}.$$

Theorem (Cole/Simpson 2006):
In $\mathcal{P}_w$ we have

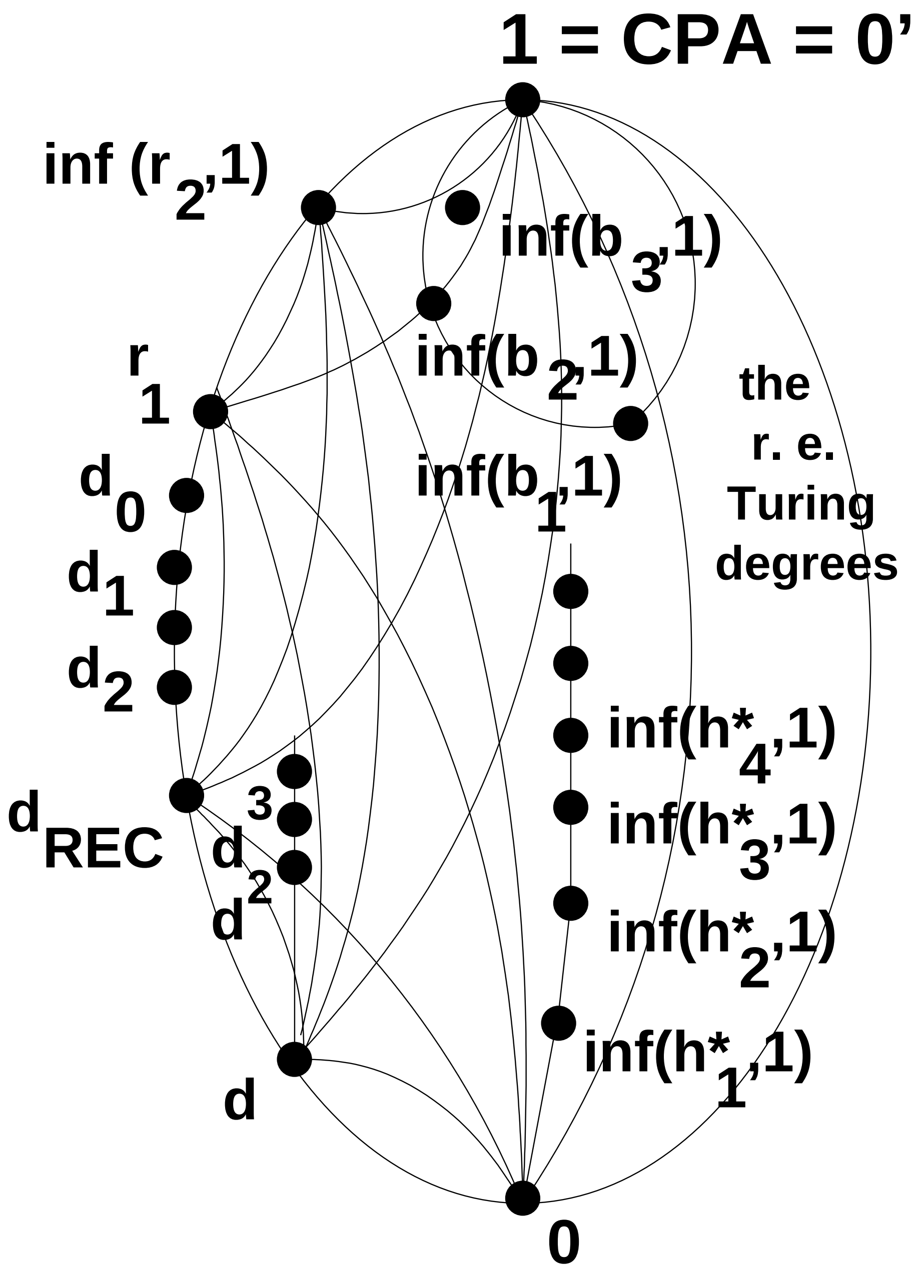
$$\mathbf{0} < \inf(\mathbf{h}^*_1, \mathbf{1}) < \inf(\mathbf{h}^*_2, \mathbf{1}) < \cdots$$

$$< \inf(\mathbf{h}^*_\alpha, \mathbf{1}) < \inf(\mathbf{h}^*_{\alpha+1}, \mathbf{1}) < \cdots < \mathbf{1}$$

and these weak degrees are incomparable with $\mathbf{d}$, $\mathbf{d}_{\mathsf{REC}}$, $\mathbf{r}_1$, $\inf(\mathbf{r}_2, \mathbf{1})$ and all recursively enumerable Turing degrees except $\mathbf{0}$ and $\mathbf{0}'$.



The weak degrees inf$(\mathbf{h}^*_\alpha, \mathbf{1})$, $1 \leq \alpha < \omega_1^{\mathsf{CK}}$, are incomparable with $\mathbf{d}$, $\mathbf{d}_{\mathsf{REC}}$, $\mathbf{r}_1$, inf$(\mathbf{r}_2, \mathbf{1})$, and all r.e. Turing degrees except $\mathbf{0}$ and $\mathbf{0}'$.



A more comprehensive picture of $\mathcal{P}_w$.
$\mathbf{r} =$ randomness, $\mathbf{h} =$ hyperarithmeticity,
$\mathbf{b} =$ almost everywhere domination,
$\mathbf{d} =$ diagonal nonrecursiveness, etc.

Definition. $S \subseteq \omega^\omega$ is Σ^0_3 if

$$S = \{f \in \omega^\omega \mid \exists i \, \forall m \, \exists n \, R(i, m, n, f)\}$$

for some recursive predicate $R \subseteq \omega^3 \times \omega^\omega$.

Many interesting mass problems are Σ^0_3.

Examples:

- R_1 is Σ^0_2.

- R_2 is Σ^0_3.

- DNR is Π^0_1.

- $\mathsf{DNR}_{\mathsf{REC}}$ is Σ^0_3.

- AED is Σ^0_3.

The Embedding Lemma:

If $S \subseteq \omega^\omega$ is Σ^0_3 and if $P \subseteq 2^\omega$ is nonempty Π^0_1, then $\deg_w(S \cup P) \in \mathcal{P}_w$.

It follows that, for many Σ^0_3 sets $S \subseteq \omega^\omega$, $\deg_w(S) \in \mathcal{P}_w$.

Examples:

1. $\mathsf{R}_1 = \{X \in 2^\omega \mid X \text{ is 1-random}\}$.

Since R_1 is Σ^0_2, it follows by the Embedding Lemma that $\mathbf{r}_1 = \deg_w(R_1) \in \mathcal{P}_w$.

2. $\mathsf{R}_2 = \{X \in 2^\omega \mid X \text{ is 2-random}\}$.

Since R_2 is Σ^0_3, it follows that $\inf(\mathbf{r}_2, \mathbf{1}) = \deg_w(R_2 \cup \mathsf{CPA}) \in \mathcal{P}_w$.

3. $\mathsf{D} = \{f \in \omega^\omega \mid f \text{ is diagonally nonrecursive}\}$.

Since D is Π^0_1, $\mathbf{d} = \deg_w(D) \in \mathcal{P}_w$.

4. $\mathsf{D}_{\mathsf{REC}} = \{f \in D \mid f \text{ is recursively dominated}\}$.

Since $\mathsf{D}_{\mathsf{REC}}$ is Σ^0_3, $\mathbf{d}_{\mathsf{REC}} = \deg_w(\mathsf{D}_{\mathsf{REC}}) \in \mathcal{P}_w$.

5. Let $A \subseteq \omega$ be r.e. Since $\{A\}$ is Π^0_2, $\deg_w(\{A\} \cup \mathsf{CPA}) \in \mathcal{P}_w$. This gives our embedding of $\mathcal{E}_T$ into $\mathcal{P}_w$.

The Embedding Lemma (restated)**:**

Let $S \subseteq \omega^\omega$ be $\mathbf{\Sigma}^0_3$. Let $P \subseteq 2^\omega$ be nonempty Π^0_1. Then $\exists$ nonempty Π^0_1 $Q \subseteq 2^\omega$ such that $Q \equiv_w S \cup P$.

Proof (sketch). **Step 1.** By Skolem functions, we may assume that $S \subseteq \omega^\omega$ is Π^0_1.

Step 2. We have $S = \{\text{paths through } T_S\}$, $P = \{\text{paths through } T_P\}$, where T_S, T_P are recursive subtrees of $\omega^{<\omega}$, $2^{<\omega}$ respectively. May assume $\tau(n) \geq 2$ for all $n < |\tau|$, $\tau \in T_S$.

Define $Q = \{\text{paths through } T_Q\}$, where T_Q is the set of all $\rho \in \omega^{<\omega}$ of the form

$$\rho = \sigma_0{}^\frown\langle m_0\rangle{}^\frown\sigma_1{}^\frown\langle m_1\rangle{}^\frown\cdots{}^\frown\langle m_{k-1}\rangle{}^\frown\sigma_k$$

where

- $\sigma_0, \sigma_1, \ldots, \sigma_k \in T_P$,
- $\langle m_0, m_1, \ldots, m_{k-1}\rangle \in T_S$,
- $\rho(n) \leq \max(n, 2)$ for all $n < |\rho|$.

One can show that $Q \equiv_w S \cup P$.

Step 3. Q is Π^0_1 and recursively bounded. Hence, we can find Π^0_1 $Q^* \subseteq 2^\omega$ such that Q^* is recursively homeomorphic to Q. Done.

A corollary of the Embedding Lemma is:

Corollary: If $\mathbf{s}$ is the weak degree of a Σ^0_3 set, then $\text{inf}(\mathbf{s}, \mathbf{1}) \in \mathcal{P}_w$.

We have seen that this provides a powerful method of producing specific, natural degrees in $\mathcal{P}_w$. We now extend this method.

If $\mathbf{s}$ is the weak degree of a Σ^0_3 set S, define

$$S' = \{X' \mid X \in S\}$$

where X' is the Turing jump of X, and

$$S^* = \{Y \mid (\exists X \in S)\,(\text{BLR}(X) \subseteq \text{BLR}(Y))\}.$$

Then S' and S^* are Σ^0_3. Moreover, the weak degrees of S' and S^* depend only on the weak degree of S.

Thus we have an *internal jump operator*

$$\text{inf}(\mathbf{s}^*, \mathbf{1}) \mapsto \text{inf}(\mathbf{s}^{*\prime *}, \mathbf{1})$$

within $\mathcal{P}_w$. The Cole/Simpson embedding of the hyperarithmetical hierarchy into $\mathcal{P}_w$ may be viewed as iterating the internal jump operator through the ordinals $< \omega_1^{\text{CK}}$.

Summary:

In this talk I have emphasized the many specific, natural, weak degrees which belong to $\mathcal{P}_w$ and are related to foundationally interesting topics:

- algorithmic randomness
- reverse mathematics
- hyperarithmeticity
- almost everywhere domination
- diagonal nonrecursiveness
- subrecursive hierarchies
- resource-bounded computational complexity
- Kolmogorov complexity
- effective Hausdorff dimension

These examples have been developed over several years, from 1999 to the present.

Recent Aspects:

Two newer aspects of $\mathcal{P}_w$ are:

1. the use of $\mathcal{P}_w$ to classify 2-dimensional dynamical systems of finite type (Simpson 2007).

2. a negative result concerning the possible use of $\mathcal{P}_w$ as a model of intuitionistic propositional calculus (Simpson 2007).

This refers to the original motivation for mass problems going back to papers of Kolmogorov 1932, Medvedev 1955, and Muchnik 1963.

These aspects were explained in my talk

Recent Aspects of Mass Problems

at FRG workshop, Chicago, September 2007.

Still more recently I have been exploring connections between mass problems and *realizability* in the sense of Kleene/Vesley.

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