Predicativity: The Outer Limits

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S. Feferman, Systems of Predicative Analysis, *J. of Symbolic Logic*, 29, 1964, pp. 1–30.

S. Feferman, Systems of Predicative Analysis, II: Representations of Ordinals, *J. of Symbolic Logic*, 33, 1968, pp. 193–220.

> "Although we strongly believe that the explications proposed in this paper for the notion of predicative provability in analysis are correct, we are not convinced that the matter has been settled conclusively by the results obtained so far. It is premature to say just what would constitute final evidence concerning this question. We expect that this will be revealed, at least in part, by further study of the theories considered here." (page 29)

Subsequent papers do not indicate that Feferman has changed his mind about predicative provability.

S. Feferman, Predicatively reducible systems of set theory, in *Axiomatic Set Theory*, Proc. Symp. Pure Math. vol. XIII, Part 2, Amer. Math. Soc., Providence, 1974, pp. 11-32.

S. Feferman, A more perspicuous formal system for predicativity, in *Konstruktionen versus Positionen, I*, Walter de Gruyter, Berlin, 1979, pp. 68–93.

S. Feferman, Reflecting on incompleteness, *J. of Symbolic Logic*, 56, 1991, pp. 1–49.

Rules versus axioms.

Δ^1_1 comprehension rule:

$$\frac{\forall n\,((\exists X\,\alpha(n,X)) \leftrightarrow (\forall Y\,\beta(n,Y)))}{\exists Z\,\forall n\,(n \in Z \leftrightarrow \exists X\,\alpha(n,X))}$$

Δ^1_1 comprehension axiom:

$$(\forall n\,((\exists X\,\alpha(n,X)) \leftrightarrow (\forall Y\,\beta(n,Y)))) \\ \rightarrow \exists Z\,\forall n\,(n \in Z \leftrightarrow \exists X\,\alpha(n,X))$$

Here α and β are arithmetical formulas.

Rules versus axioms.

hierarchy rule:

$$\frac{WO(<_e)}{\forall X\,\exists Y\,H(<_e, X, Y)}$$

hierarchy axiom:

$$\forall Z\,(WO(Z) \rightarrow \forall X\,\exists Y\,H(Z, X, Y))$$

$WO(Z)$: Z is a well ordering.

$H(Z, X, Y)$: Y is a Turing jump hierararchy along Z starting at X.

Rules versus axioms.

transfinite induction rule:

$$\frac{WO(<_e)}{TI(<_e,\gamma),\ \gamma \text{ arbitrary}}$$

transfinite induction axiom:

$$\forall Z\,(WO(Z) \to TI(Z,\gamma))$$

for arbitrary γ

$TI(Z,\gamma)$: transfinite induction along Z with respect to the formula γ.

IR consists of the Δ^1_1 comprehension rule + the hierarchy rule + the transfinite induction rule.

ATR_0 consists of the hierarchy axiom. It includes the Δ^1_1 comprehension axiom. It is a system with restricted induction and so does not include the transfinite induction rule.

$|\mathsf{IR}| = |\mathsf{ATR}_0| = \Gamma_0$.

IR and ATR_0 prove the same Π^1_1 sentences.

IR and ATR_0 have the same proof-theoretic strength.

THEMES OF THIS TALK:

1. IR explicates predicative provability, while ATR_0 explicates predicative reducibility.

2. ATR_0 is much stronger than IR, model-theoretically and, above all, mathematically.

The minimum ω-model of IR is HYP(Γ_0).

The minimum ω-model of the Δ^1_1 comprehension axiom is HYP, i.e., $L_{\omega_1^{CK}} \cap P(\omega)$.

HYP is the intersection of all ω-models of ATR_0.

ATR_0 has no minimal ω-model.

ATR_0 holds in any β-model.

HYP is the intersection of all β-models.

There is no minimal β-model.

A set-theoretic version of ATR_0.

$\mathsf{ATR}_0^{\mathsf{set}}$ = extensionality

+ foundation axiom:
$\forall x\,(x \neq \emptyset \rightarrow \exists u \in x\,(u \cap x = \emptyset))$

+ closure under F_0–F_8
(rudimentary functions)

+ axiom of infinity

+ $\forall x\,(x$ is hereditarily countable)

+ axiom beta:
$\forall r\,(WF(r) \rightarrow \exists f\,(\mathsf{field}(r) \subseteq \mathsf{dom}(f)$
$\wedge \forall u \in \mathsf{dom}(f)\,(f(u) = \{f(v) : \langle v, u\rangle \in r\})))\,.$

Theorem (Simpson). $\mathsf{ATR}_0^{\mathsf{set}}$ is conservative over ATR_0. Actually, it is a definitional extension of ATR_0, where well founded trees encode hereditarily countable sets in the usual way.

Π^1_∞-TI_0 consists of the transfinite induction axiom. It includes ATR_0. Precisely, Σ^1_1-$\mathsf{TI}_0 = \mathsf{ATR}_0 + \Sigma^1_1$-$\mathsf{IND}$.

Π^1_∞-TI_0 is proof-theoretically stronger than IR and ATR_0.

$|\Pi^1_\infty$-$\mathsf{TI}_0| = \varphi_{\varepsilon_{\Omega+1}}(0) =$ the Howard ordinal.

Π^1_∞-TI_0 has no minimal ω-model.

Π^1_∞-TI_0 holds in any β-model.

HYP is the intersection of all β-models.

There is no minimal β-model.

Reverse mathematics of ATR_0:

The following are equivalent over RCA_0.

1. ATR_0.

2. Every disjoint pair of analytic sets can be separated by a Borel set.

3. The domain of a single-valued Borel set in the plane is Borel.

4. Every uncountable closed (or analytic) set has a perfect subset.

5. Clopen (or open) determinacy.

6. Clopen (or open) Ramsey theorem.

7. For every countable bipartite graph, there exists a *König covering*, i.e., a pair (C, M) where C is a vertex covering, M is a matching, and C consists of one vertex of each edge in M.

   This is a combined result of Aharoni/Magidor/Shore 1992 and Simpson 1994.

8. Comparability of countable well orderings.

9. Ulm theory for countable reduced Abelian p-groups.

Open questions:

1. Is Fraïssé's conjecture for countable linear orderings provable in ATR_0?

   See papers by Marcone 1994 and Shore 1994.

2. A consequence of the Ulm theory:

   > If each of two countable reduced Abelian p-groups is a direct summand of the other, then they are isomorphic.

   Is this statement equivalent to ATR_0?

Note: Mathematically, IR seems no stronger than arithmetical comprehension, and Π^1_∞-TI_0 no stronger than ATR_0.

Aharoni/Magidor/Shore, On the strength of König's duality theorem for countable bipartite graphs, *J. of Combinatorial Theory*, series B, 54, 1992, pp. 257–290.

Simpson, same title, *J. of Symbolic Logic*, 59, 1994, pp. 113–123.

R. Shore, On the strength of Fraïssé's conjecture, in *Logical Methods*, Birkhäuser, 1993, pp. 782–813.

A. Marcone, Foundations of BQO theory, Transactions of the AMS, 345, 1994, pp. 641–660.

My book on reverse mathematics is finally out!

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Perspectives in Mathematical Logic
Springer-Verlag, 1998
XIV + 445 pages.

Order: 1-800-SPRINGER.

List price: $60.

Discount: 30 percent for ASL members; mention promotion code S206.

Impredicative Π^0_2 combinatorial theorems.

1. A finite miniaturization (a la Kirby/Paris) of the clopen Ramsey theorem:

Friedman/McAloon/Simpson, A finite combinatorial principle which is equivalent to the 1-consistency of predicative analysis, in: *Logic Symposion I (Patras)*, edited by G. Metakides, North-Holland, Amsterdam, 1982, pp. 197–220.

2. Friedman's work on Kruskal's theorem:

Stephen G. Simpson, Nonprovability of certain combinatorial properties of finite trees, in: *Harvey Friedman's Research in the Foundations of Mathematics*, North-Holland, Amsterdam, 1985, pp. 87–117.

Definitions.

1. Let X be a finite set of positive integers. A *coloring* of X is $P(X) = C_1 \cup C_2$ where C_1 and C_2 are closed under initial segment. $Y \subseteq X$ is *homogeneous* if $P(Y) \subseteq C_1$ or $P(Y) \subseteq C_2$.

2. X is 0-*dense* if $|X| \geq 2$ and $|X| \geq \min X$.

3. X is $n+1$-*dense* if for every coloring of X there exists an n-dense homogeneous set.

Theorem (Friedman/McAloon/Simpson).

$\forall\, n\ \exists\, n$-dense set
$\equiv$ uniform Π^0_2 reflection for IR
$\equiv$ uniform Π^0_2 reflection for ATR_0

3. A recent result of Friedman:

Definitions. A *tree* T is a finite poset with a minimum element such that the predecessors of each element are linearly ordered. The *height* of $x \in T$ is the number of predecessors of x in T. The *height of* T is the maximum height of an element of T. We say that T is *of degree* $\leq k$ if each element of T has at most k immediate successors.

$T(\leq i) = \{x \in T : \text{height}(x) \leq i\}$.

$T(i) = \{x \in T : \text{height}(x) = i\}$.

$T(> i) = \{x \in T : \text{height}(x) > i\}$.

Note that $T(\leq i)$ is a subtree of T.

Consider the following statement.

> For each k there exists n so large that the following holds. If T is a tree of height n and degree $\leq k$, then there exists $1 \leq i \leq n$ and an inf-preserving embedding of $T(\leq i)$ into T which carries $T(i)$ into $T(> i)$.

Friedman 1998 showed that this statement is equivalent to uniform Π^0_2 reflection for Π^1_2-TI_0.