PREDICATIVITY: THE OUTER LIMITS

STEPHEN G. SIMPSON

Abstract. Beginning with ideas of Poincaré and Weyl, Feferman in the sixties undertook a profound analysis of the predicativist foundational program. He presented a subystem of second order arithmetic IR and argued convincingly that it represents the outer limits of what is predicatively provable. Much later, Friedman introduced another system ATR_0 which is conservative over IR for Π^1_1 sentences yet includes several well known theorems of algebra, descriptive set theory, and countable combinatorics that are not provable in IR. The proof-theoretic ordinal of both systems is Γ_0. ATR_0 has emerged as one of a handful of systems that are important for reverse mathematics. From a foundational standpoint, we may say that IR represents predicative provability while ATR_0 represents predicative reducibility. Subsequently Friedman formulated mathematically natural finite combinatorial theorems that are not only not predicatively provable but go beyond Γ_0 and therefore are not predicatively reducible.

§1. IR and ATR_0. In his first major work on systems of predicative analysis [3, 4], Feferman introduces the system IR and proposes it as an explication of predicative provability.

> "Although we strongly believe that the explications proposed in this paper for the notion of predicative provability in analysis are correct, we are not convinced that the matter has been settled conclusively by the results obtained so far. It is premature to say just what would constitute final evidence concerning this question. We expect that this will be revealed, at least in part, by further study of the theories considered here." (page 29)

In subsequent papers on predicative provability, Feferman does not back away from this proposal. The systems that he introduces in [5, 6, 7] as explications of predicative provability are conservative over IR.

Received by the editors October 29, 1999 and, in revised form, October 16, 2000.

1991 *Mathematics Subject Classification.* 03F35, 03D80, 03B30.

Preparation of this paper was partially supported by NSF grant DMS-0070718.

This paper is a writeup of my talk at a symposium at Stanford University, December 11–13, 1998, in honor of Solomon Feferman's 70th birthday. I would like to thank Sol for his encouragement over many years and especially when I was writing my book [26] on reverse mathematics and subsystems of second order arithmetic.

Meeting

In this paper I want to compare Feferman's system IR [3, 4] with another subsystem of second order arithmetic ATR_0 due to Friedman [10]. We begin by briefly reviewing the definitions of these systems.

§2. Rules Versus Axioms. Both IR and ATR_0 are formal systems or theories in the language of second order arithmetic. While ATR_0 is easily defined by means of a finite set of axioms, IR is more conveniently described in terms of inference rules rather than axioms.

1. The Δ^1_1 Comprehension Axiom:

$$(\forall n\,((\exists X\,\alpha(n,X)) \leftrightarrow (\forall Y\,\beta(n,Y))))$$
$$\rightarrow \exists Z\,\forall n\,(n \in Z \leftrightarrow \exists X\,\alpha(n,X))$$

   Here α and β are arithmetical formulas.

2. The Δ^1_1 Comprehension Rule:

$$\frac{\forall n\,((\exists X\,\alpha(n,X)) \leftrightarrow (\forall Y\,\beta(n,Y)))}{\exists Z\,\forall n\,(n \in Z \leftrightarrow \exists X\,\alpha(n,X))}$$

   Here α and β are as above.

3. The Hierarchy Axiom:

$$\forall Z\,(\mathrm{WO}(Z) \rightarrow \forall X\,\exists Y\,\mathrm{H}(Z,X,Y))$$

   Here $\mathrm{WO}(Z)$ is a Π^1_1 formula expressing that Z is a well ordering of the integers, and $\mathrm{H}(Z,X,Y)$ is an arithmetical formula expressing that Y is a Turing jump hierararchy along Z starting at X.

4. The Hierarchy Rule:

$$\frac{\mathrm{WO}(<_e)}{\forall X\,\exists Y\,\mathrm{H}(<_e,X,Y)}$$

   Here $<_e$ is a primitive recursive linear ordering of the integers, and $\mathrm{WO}(Z)$ and $\mathrm{H}(Z,X,Y)$ are as above.

5. The Transfinite Induction Axiom:

$$\forall Z\,(\mathrm{WO}(Z) \rightarrow \mathrm{TI}(Z,\gamma))$$

   Here $\mathrm{WO}(Z)$ is as above, γ is an arbitrary formula, and $\mathrm{TI}(Z,\gamma)$ expresses transfinite induction along Z with respect to γ.

6. The Transfinite Induction Rule:

$$\frac{\mathrm{WO}(<_e)}{\mathrm{TI}(<_e,\gamma)}$$

   Here $<_e$ and $\mathrm{WO}(Z)$ and $\mathrm{TI}(Z,\gamma)$ are as above.

We are now ready to define the systems IR and ATR_0.

1. IR consists of the Δ^1_1 Comprehension Rule, the Hierarchy Rule, and the Transfinite Induction Rule.
2. ATR_0 consists of the Hierarchy Axiom. It is known that ATR_0 includes the Δ^1_1 Comprehension Axiom. It is a system with restricted induction (see Friedman [10]) and so does not include the Transfinite Induction Rule.

§3. Model-Theoretic Properties of IR and ATR_0. It is known that IR and ATR_0 are proof-theoretically similar:

1. They have the same proof-theoretic ordinal: $|\mathsf{IR}| = |\mathsf{ATR}_0| = \Gamma_0$.
2. IR and ATR_0 prove the same Π^1_1 sentences. In particular, they prove the same arithmetical sentences.
3. IR and ATR_0 have the same proof-theoretic strength.

These results are due to Friedman [11, §4]. The main point that we would like to make here is that IR and ATR_0 differ greatly in some other, very significant respects. In particular:

1. IR explicates predicative provability, while ATR_0 explicates predicative reducibility.
2. ATR_0 is much stronger than IR, model-theoretically and, above all, mathematically.

The following properties of the two systems indicate how different they are from the model-theoretic point of view.

1. The minimum ω-model of IR is $\mathrm{HYP}(\Gamma_0)$, i.e., $L_{\Gamma_0} \cap P(\omega)$. This is a relatively small initial segment of
$$\mathrm{HYP} = \{X \subseteq \omega : X \text{ is hyperarithmetical}\},$$
i.e., $L_{\omega_1^{CK}} \cap P(\omega)$.
2. The minimum ω-model of the Δ^1_1 Comprehension Axiom is HYP.
3. HYP is the intersection of all ω-models of ATR_0.
4. ATR_0 has no minimal ω-model.
5. ATR_0 automatically holds in any β-model.
6. HYP is the intersection of all β-models.
7. There is no minimal β-model.

We can also compare IR and ATR_0 with the perhaps more familiar system $\Pi^1_\infty\text{-}\mathsf{TI}_0$ consisting of the Transfinite Induction Axiom. The latter system is sometimes known as *bar induction*. Some model-theoretic properties:

1. $\Pi^1_\infty\text{-}\mathsf{TI}_0$ includes both IR and ATR_0. The precise relationship to ATR_0 is that $\Sigma^1_1\text{-}\mathsf{TI}_0 = \mathsf{ATR}_0 + \Sigma^1_1\text{-}\mathsf{IND}$ (Simpson [22]).
2. $\Pi^1_\infty\text{-}\mathsf{TI}_0$ is proof-theoretically stronger than IR and ATR_0.
3. $|\Pi^1_\infty\text{-}\mathsf{TI}_0| = \varphi_{\varepsilon_{\Omega+1}}(0) =$ the Howard ordinal.
4. $\Pi^1_\infty\text{-}\mathsf{TI}_0$ has no minimal ω-model.
5. $\Pi^1_\infty\text{-}\mathsf{TI}_0$ automatically holds in any β-model.
6. HYP is the intersection of all β-models.

7. There is no minimal β-model.

For proofs of the model-theoretic results mentioned above, see Chapters VII and VIII of Simpson [26].

REMARK 1. We see above that ATR_0 and Π^1_∞-TI_0 are model-theoretically stronger than IR, in that the ω-models are larger. One might think that the greater strength comes from the fact that ATR_0 and Π^1_∞-TI_0 deal with arbitrary well orderings of the integers, and not only primitive recursive ones. However, this is not the case. Letting ATR_0^- be ATR_0 with the Hierarchy Axiom restricted to primitive recursive linear orderings, the above model-theoretic results for ATR_0 continue to hold for ATR_0^-. Thus these results are seen to have a certain robustness.

§4. A Set-Theoretic Version of ATR_0. In [5] Feferman introduced a set-theoretic version of IR. Subsequently Simpson [21] introduced a set-theoretic version of ATR_0 known as $\mathsf{ATR}_0^{\text{set}}$, defined by

$$
\begin{aligned}
\mathsf{ATR}_0^{\text{set}} \quad = \quad & \text{Axiom of Extensionality} \\
+ \quad & \text{Axiom of Foundation:} \\
& \quad \forall x\,(x \neq \emptyset \rightarrow \exists u \in x\,(u \cap x = \emptyset)) \\
+ \quad & \text{closure under } F_0\text{–}F_8\text{, i.e., under rudimentary functions} \\
+ \quad & \text{Axiom of Infinity} \\
+ \quad & \forall x\,(x \text{ is hereditarily countable}) \\
+ \quad & \text{Axiom Beta:} \\
& \quad \forall r\,(\mathrm{WF}(r) \rightarrow \exists f\,(\mathrm{field}(r) \subseteq \mathrm{dom}(f) \\
& \qquad \wedge\ \forall u \in \mathrm{dom}(f)\,(f(u) = \{f(v) : \langle v, u\rangle \in r\})))\,.
\end{aligned}
$$

It is shown in Simpson [21] (see also [26, §VII.3]) that $\mathsf{ATR}_0^{\text{set}}$ is conservative over ATR_0. Actually, it is a definitional extension of ATR_0, where well founded trees encode hereditarily countable sets in the usual way.

§5. Reverse Mathematics of ATR_0. As is well known, ATR_0 is one of the five basic systems of reverse mathematics. From Simpson [26, Chapter V] we have:

THEOREM 2. The following are equivalent over RCA_0.

1. ATR_0.
2. Every disjoint pair of analytic sets can be separated by a Borel set.
3. The domain of a single-valued Borel set in the plane is Borel.
4. Every uncountable closed (or analytic) set has a perfect subset.
5. Clopen (or open) determinacy.

6. The clopen (or open) Ramsey Theorem.
7. For every countable bipartite graph, there exists a *König covering*, i.e., a pair (C, M) where C is a vertex covering, M is a matching, and C consists of one vertex of each edge in M. (This is a combined result of Aharoni/Magidor/Shore 1992 [1] and Simpson 1994 [25].)
8. Comparability of countable well orderings, i.e., well orderings of the integers.
9. The Ulm theory for countable reduced Abelian p-groups.

There are some interesting open questions concerning the reverse mathematics aspect of ATR_0.

1. Is Fraïssé's conjecture for countable linear orderings provable in ATR_0? See [26, X.3.31] and Marcone 1994 [16] and Shore 1993 [19].
2. A well known consequence of the Ulm theory (see Kaplansky [14]) is:

   If each of two countable reduced Abelian p-groups is a direct summand of the other, then they are isomorphic.

   Is this statement equivalent to ATR_0? This question is due to Friedman (see [26, V.7.7]). Some recent progress on this question is in Friedman [9].

Remark 3. The reverse mathematics investigations of [26, Chapter V] seem to indicate that, mathematically, IR is no stronger than ACA_0 (arithmetical comprehension), and Π^1_∞-TI_0 is no stronger than ATR_0. Thus ATR_0 is a much better system than IR from the viewpoint of reverse mathematics.

§6. Impredicative Π^0_2 Combinatorial Theorems. In the aftermath of Paris/Harrington, it has been shown that certain mathematically appealing, finite combinatorial theorems are not provable in IR or ATR_0 or even stronger systems. In particular, such theorems are neither predicatively provable nor predicatively reducible. For a survey of this general area, see Simpson [24]. Recently Feferman [8] has cited some of these results in footnotes. We now state some of these results.

6.1. An Impredicative Ramsey-Type Theorem. Friedman/McAloon/Simpson 1982 [11] were the first to exhibit a mathematically natural, finite combinatorial theorem which is not predicatively provable. To state their result, we need some definitions.

Definition 4. Let X be a finite set of positive integers.

1. A *coloring* of X is given by $P(X) = C_1 \cup C_2$, where C_1 and C_2 are closed under initial segment. A set $Y \subseteq X$ is said to be *homogeneous* for the given coloring if either $P(Y) \subseteq C_1$ or $P(Y) \subseteq C_2$.
2. X is said to be 0-*dense* if $|X| \geq 2$ and $|X| \geq \min X$. X is said to be $(n{+}1)$-*dense* if for every coloring of X there exists an n-dense homogeneous set.

Theorem 5 (Friedman/McAloon/Simpson). The following statements are pairwise equivalent over PRA.

1. $\forall n\ \exists n$-dense finite set.
2. uniform Π^0_2 reflection for IR.
3. uniform Π^0_2 reflection for ATR_0.

6.2. Friedman's Work on Kruskal's Theorem. Friedman has shown that certain interesting combinatorial properties of finite trees are not provable in IR and ATR_0 and stronger systems. See the exposition in Simpson [23].

It turns out that there is a close connection between these results and recent spectacular work in finite graph theory. Namely, Friedman/Robertson/Seymour [12] have used them to show that the celebrated Robertson/Seymour Graph Minor Theorem is not provable in Π^1_1-CA_0.

6.3. A Recent Result of Friedman. In order to state Friedman's most recent result along these lines, we first give the necessary definitions.

DEFINITION 6.

1. A *tree* T is a finite poset with a minimum element such that the predecessors of each element are linearly ordered. The *height* of $x \in T$ is the number of predecessors of x in T. The *height of* T is the maximum height of an element of T. We say that T is *of degree* $\leq k$ if each element of T has at most k immediate successors.
2. $T(\leq i) = \{x \in T : \text{height}(x) \leq i\}$.
3. $T(i) = \{x \in T : \text{height}(x) = i\}$.
4. $T(> i) = \{x \in T : \text{height}(x) > i\}$.

Note that $T(\leq i)$ is a subtree of T.

Now consider the following combinatorial statement concerning finite trees.

> For each k there exists n so large that the following holds. If T is a tree of height n and degree $\leq k$, then there exists $1 \leq i \leq n$ and an inf-preserving embedding of $T(\leq i)$ into T which carries $T(i)$ into $T(> i)$.

Friedman has recently shown that this statement is equivalent to uniform Π^0_2 reflection for Π^1_2-TI_0. In particular, this statement of Friedman is true but not provable in ATR_0. See also Friedman's contribution to this volume.

REFERENCES

[1] RON AHARONI, MENACHEM MAGIDOR, and RICHARD A. SHORE, *On the strength of König's duality theorem for infinite bipartite graphs*, ***Journal of Combinatorial Theory, Series B***, vol. 54 (1992), pp. 257–290.

[2] J. N. Crossley, J. B. Remmel, R. A. Shore, and M. E. Sweedler (editors), ***Logical Methods***, Birkhäuser, 1993, 813 pages.

[3] SOLOMON FEFERMAN, *Systems of predicative analysis, I*, ***The Journal of Symbolic Logic***, vol. 29 (1964), pp. 1–30.

[4] ———, *Systems of predicative analysis, II*, ***The Journal of Symbolic Logic***, vol. 33 (1968), pp. 193–220.

[5] ———, *Predicatively reducible systems of set theory*, [18, part 2], 1974, pp. 11–32.

[6] ———, *A more perspicuous formal system for predicativity*, [15, volume i], 1978, pp. 68–93.

[7] ———, *Reflecting on incompleteness*, ***The Journal of Symbolic Logic***, vol. 56 (1991), pp. 1–49.

[8] ———, *Does mathematics need new axioms?*, ***American Mathematical Monthly***, vol. 106 (1999), pp. 99–111.

[9] Harvey Friedman, *Metamathematics of Ulm theory*, June 24, 1999, unpublished, 34 pages.

[10] ———, *Systems of second order arithmetic with restricted induction, I, II* (abstracts), ***The Journal of Symbolic Logic***, vol. 41 (1976), pp. 557–559.

[11] Harvey Friedman, Kenneth McAloon, and Stephen G. Simpson, *A finite combinatorial principle which is equivalent to the 1-consistency of predicative analysis*, [17], 1982, pp. 197–230.

[12] Harvey Friedman, Neil Robertson, and Paul Seymour, *The metamathematics of the graph minor theorem*, [20], 1987, pp. 229–261.

[13] L. A. Harrington, M. Morley, A. Scedrov, and S. G. Simpson (editors), ***Harvey Friedman's Research on the Foundations of Mathematics***, Studies in Logic and the Foundations of Mathematics, North-Holland, 1985, XVI + 408 pages.

[14] Irving Kaplansky, ***Infinite Abelian Groups***, revised ed., University of Michigan Press, 1969, VII + 95 pages.

[15] K. Lorenz (editor), ***Konstruktionen versus Positionen, beiträge zur Diskussion um die Konstruktive Wissenschaftstheorie***, Walter de Gruyter, Berlin, 1978, Volume I, XX + 350 pages, Volume II, X + 406 pages.

[16] Alberto Marcone, *Foundations of BQO theory*, ***Transactions of the American Mathematical Society***, vol. 345 (1994), pp. 641–660.

[17] G. Metakides (editor), ***Patras Logic Symposion***, Studies in Logic and the Foundations of Mathematics, North-Holland, 1982, IX + 391 pages.

[18] D. S. Scott and T. J. Jech (editors), ***Axiomatic Set Theory***, Proceedings of Symposia in Pure Mathematics, vol. XIII, American Mathematical Society, 1971–1974, Part 1, VI + 474 pages, Part 2, VI + 222 pages.

[19] Richard A. Shore, *On the strength of Fraïssé's conjecture*, [2], 1993, pp. 782–813.

[20] S. G. Simpson (editor), ***Logic and Combinatorics***, Contemporary Mathematics, American Mathematical Society, 1987, XI + 394 pages.

[21] Stephen G. Simpson, *Set theoretic aspects of* ATR_0, [27], 1982, pp. 255–271.

[22] ———, Σ^1_1 *and* Π^1_1 *transfinite induction*, [27], 1982, pp. 239–253.

[23] ———, *Nonprovability of certain combinatorial properties of finite trees*, [13], 1985, pp. 87–117.

[24] ———, *Unprovable theorems and fast growing functions*, [20], 1987, pp. 359–394.

[25] ———, *On the strength of König's duality theorem for countable bipartite graphs*, ***The Journal of Symbolic Logic***, vol. 59 (1994), pp. 113–123.

[26] ———, ***Subsystems of Second Order Arithmetic***, Perspectives in Mathematical Logic, Springer-Verlag, 1999, XIV + 445 pages.

[27] D. van Dalen, D. Lascar, and T. J. Smiley (editors), ***Logic Colloquium '80***, Studies in Logic and the Foundations of Mathematics, North-Holland, 1982, X + 342 pages.

E-mail: `simpson@math.psu.edu`
URL: `http://www.math.psu.edu/simpson/`

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, STATE COLLEGE, PA 16802, USA