

A survey of basis theorems

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Basis theorems.

A basis theorem is a theorem of the form:

For any nonempty effectively closed set in Euclidean space, at least one member of the set is “close to being computable”.

Some well known basis theorems are:

- the Low Basis Theorem,
- the R.E. Basis Theorem,
- the Hyperimmune-Free Basis Theorem,
- the Cone Avoidance Basis Theorem,
- the Randomness Preservation Basis Thm.

Less well known is a basis theorem of Higuchi/Hudelson/Simpson/Yokoyama on preservation of partial randomness.

We shall state these basis theorems, discuss some of their applications, and discuss the possibilities for combining them.

Three basis theorems.

Let $\leq_T$ denote Turing reducibility.

Let $'$ denote the Turing jump operator.

The Low Basis Theorem:

For any nonempty effectively closed set Q , there exists $Z \in Q$ such that $Z' \leq_T 0'$.

The R.E. Basis Theorem:

For any nonempty effectively closed set Q , there exists $Z \in Q$ such that Z is of recursively enumerable Turing degree.

We say that Z is *hyperimmune-free* if $(\forall \text{ functions } f \leq_T Z) (\exists \text{ recursive function } g) \forall n (f(n) < g(n))$.

The Hyperimmune-Free Basis Theorem:

For any nonempty effectively closed set Q , $(\exists Z \in Q) (Z \text{ is hyperimmune-free})$.

These three basis theorems are due to Jockusch/Soare 1972.

Some applications.

Basis theorems are applicable to the study of models of first- and second-order arithmetic.

Namely, there is a nonempty effectively closed set Q_ω such that each $Z \in Q_\omega$ encodes a countable ω -model of WKL_0 (subsystems of second-order arithmetic), or equivalently, a Scott set (models of first-order arithmetic).

Thus, there exist $Z_1, Z_2, Z_3 \in Q_\omega$ such that Z_1 is low, Z_2 is of r.e. Turing degree, and Z_3 is hyperimmune-free.

Question: Does there exist $Z \in Q_\omega$ with two or more of these properties?

Answer: See the next slide.

Conversely, for any ω -model M of WKL_0 and any nonempty effectively closed set Q , we have $M \cap Q \neq \emptyset$.

Thus Q_ω is in a sense universal.

For many purposes, we may assume $Q = Q_\omega$.

Can we combine these basis theorems?

No. The Jockusch/Soare basis theorems are known to be “pairwise incompatible.”

1. The Arslanov Completeness Criterion provides a nonempty effectively closed Q such that for all r.e. sets A , if $(\exists Z \in Q) (Z \leq_T A)$ then $0' \leq_T A$.

Therefore, the Low Basis Theorem and the R.E. Basis Theorem cannot be combined into one basis theorem.

2. It is known that for hyperimmune-free Z one cannot have $0 <_T Z \leq_T 0'$.

Therefore, the Hyperimmune-Free Basis Theorem cannot be combined with the Low Basis Theorem or with the R.E. Basis Theorem.

Two more basis theorems.

The Cone Avoidance Basis Theorem:

For any nonempty effectively closed set Q ,
if $A \not\leq_T 0$ then $(\exists Z \in Q) (A \not\leq_T Z)$.

More generally,

if $\forall i (A_i \not\leq_T 0)$ then $(\exists Z \in Q) \forall i (A_i \not\leq_T Z)$.

Gandy/Kreisel/Tait, 1960.

Let $\text{MLR} = \{X \mid X \text{ is Martin-Löf random}\}$.

Let $\text{MLR}^Z = \{X \mid X \text{ is Martin-Löf random relative to } Z\}$.

The Randomness Preservation Basis Theorem:

For any nonempty effectively closed set Q ,
if $X \in \text{MLR}$ then $(\exists Z \in Q) (X \in \text{MLR}^Z)$.

Reimann/Slaman, not yet published.

Downey/Hirschfeldt/Miller/Nies, 2005.

Simpson/Yokoyama, 2011.

More applications.

An application of Cone Avoidance:

Let T be a recursively axiomatizable consistent theory extending first-order arithmetic PA (or even Robinson's Q). Define the hard core of T as $HC(T) = \bigcap \{M \mid M \text{ is the Scott set of some model of } T\}$. Then $HC(T) = REC = \{A \subseteq \mathbb{N} \mid A \text{ is recursive}\}$.

In particular, REC is the intersection of all ω -models of WKL_0 .

The use of the term “hard core” in this context was suggested by Kreisel.

More applications.

Applications of Randomness Preservation:

1. (Reimann/Slaman) $X \not\leq_T 0$ implies X is non-atomically random with respect to some Borel probability measure.
2. (Simpson/Yokoyama) Given a countable ω -model M of $WWKL_0$, we can extend M to a countable ω -model M_1 of WKL_0 such that $C \cap M \neq \emptyset$ for every M_1 -coded closed set C of positive measure. This has consequences for the reverse mathematics of non-standard measure theory.
3. (Brattka/Miller/Nies) A real number x is computably random if and only if every computably continuous function of bounded variation is differentiable at x .

More combinations of basis theorems?

It is known that Cone Avoidance can be combined with the Low Basis Theorem, or with the Hyperimmune-free Basis Theorem, but not with the R.E. Basis Theorem. (See for instance Downey/Hirschfeldt §2.19.3.)

Also, Randomness Preservation cannot be combined with the Low or the R.E. or the Hyperimmune-Free Basis Theorem.

A leading question: Can Cone Avoidance be combined with Randomness Preservation?

The answer to this question involves LR-reducibility.

Define $A \leq_{LR} B \iff MLR^B \subseteq MLR^A$. Clearly $A \leq_T B$ implies $A \leq_{LR} B$, and it is known that $A \leq_{LR} 0$ implies $A' \leq_T 0'$. A major theorem of Nies is that $A \leq_{LR} 0 \iff A$ is K-trivial. See Nies 2009 or Downey/Hirschfeldt 2010.

A theorem which combines Cone Avoidance and Randomness Preservation:

Theorem 1 (Simpson/Stephan, 2013).

For any nonempty effectively closed set Q , if $X \in \text{MLR}$ and $\forall i (A_i \not\leq_{\text{LR}} 0 \text{ or } A_i \not\leq_{\text{T}} X)$, then $(\exists Z \in Q) (X \in \text{MLR}^Z \text{ and } \forall i (A_i \not\leq_{\text{T}} Z))$.

On the other hand, let $\Omega \in \text{MLR}$ be such that $\Omega \equiv_{\text{T}} 0'$. It is well known that such reals exist (Chaitin, Kučera/Gács).

Theorem 2 (Simpson/Stephan, 2013).

$\exists$ nonempty effectively closed set Q such that $(\forall A \leq_{\text{LR}} 0) (\forall Z \in Q) (\Omega \in \text{MLR}^Z \Rightarrow A \leq_{\text{T}} Z)$.

The proof uses a result of Miller 2010.

Summary of Theorems 1 and 2:

Randomness Preservation cannot be combined with Cone Avoidance, but only because $A \not\leq_{\text{T}} 0$ does not imply $A \not\leq_{\text{LR}} 0$.

Another application.

WKL_0 is a subsystem of Z_2 which is good for the reverse mathematics of compactness (Heine-Borel, Arzela-Ascoli, Hahn-Banach, fixed points, prime ideals, etc.).

$WWKL_0$ is a subsystem of WKL_0 which is good for the reverse mathematics of measure theory (countable additivity, Monotone and Dominated Convergence theorems, Vitali Covering Lemma, etc.).

Let M be a countable ω -model of $WWKL_0$.

By Simpson/Yokoyama 2011, we get a countable ω -model $M_1 \supseteq M$ of WKL_0 such that $C \cap M \neq \emptyset$ for every M_1 -coded closed set C of positive measure.

This is called a good extension of M .

As an application of Theorem 1, we get two good extensions $M_1, M_2 \supseteq M$ such that $M = M_1 \cap M_2$.

Preservation of partial randomness.

Let $f : \{0, 1\}^* \rightarrow [-\infty, \infty]$ be an arbitrary recursive function.

For $S \subseteq \{0, 1\}^*$ let $\text{wt}_f(S) = \sum_{\sigma \in S} 2^{-f(\sigma)}$,
 $\text{pwt}_f(S) = \sup\{\text{wt}_f(P) \mid P \subseteq S \text{ prefix-free}\}$,
and $\llbracket S \rrbracket = \{X \in \{0, 1\}^{\mathbb{N}} \mid (\exists \sigma \in S) (\sigma \subset X)\}$.

We say that X is strongly f -random if
 $X \notin \bigcap_n \llbracket S_n \rrbracket$ for all uniformly r.e. $S_n \subseteq \{0, 1\}^*$
such that $\forall n (\text{pwt}_f(S_n) \leq 2^{-n})$.

Martin-Löf randomness is the special case
 $f(\sigma) = |\sigma|$. In this case $\text{pwt}_f(S) = \mu(\llbracket S \rrbracket)$
where μ is the fair coin measure on $\{0, 1\}^{\mathbb{N}}$.

Partial Randomness Preservation:

For any nonempty effectively closed set Q ,
if X is strongly f -random then $(\exists Z \in Q)$
 $(X \text{ is strongly } f\text{-random relative to } Z)$.

More generally, if $\forall i (X_i \text{ is strongly } f_i\text{-random})$
then $(\exists Z \in Q) \forall i (X_i \text{ is strongly } f_i\text{-random}$
relative to $Z)$.

Higuchi/Hudelson/Simpson/Yokoyama, 2012.

To what extent can we combine
Partial Randomness Preservation
with Cone Avoidance?

Theorem 3 (implicit in H/H/S/Y 2012).
For any nonempty effectively closed set Q ,
if $\forall i (A_i \not\leq_{LR} 0$ and X_i is strongly f_i -random),
then $(\exists Z \in Q) \forall i (A_i \not\leq_{LR} Z$ and X_i is strongly
 f_i -random relative to Z).

On the other hand, because of Theorem 2,
we cannot always replace $\leq_{LR}$ by $\leq_T$.

Can we sometimes replace $\leq_{LR}$ by $\leq_T$?

A typical open question:

Define X to be strongly half-random $\iff$
 X is strongly f -random where $f(\sigma) = |\sigma|/2$.

If Q is nonempty effectively closed, and
if $A \not\leq_T 0$ and X is strongly half-random,
does there exist $Z \in Q$ such that $A \not\leq_T Z$
and X is strongly half-random relative to Z ?

Proofs of Theorems 1 and 2.

To prove Theorem 1, we use the Cone Avoidance Basis Theorem, relativized to X .

To prove Theorem 2, we use K = prefix-free Kolmogorov complexity.

(1) If $\Omega \in \text{MLR}^Z$ then $|K(n) - K^Z(n)| \leq O(1)$ for infinitely many n . (Miller 2010.)

(2) If $\Omega \in \text{MLR}^Z$ and $Z \in Q_\omega$ then $\exists$ an infinite Z -recursive set A and a Z -recursive function $\tilde{K}$ such that $|K(n) - \tilde{K}(n)| \leq O(1)$ for all $n \in A$.

(3) Let C = plain Kolmogorov complexity. Chaitin 1976 proved: every C -trivial real is computable. Using $\tilde{K}$ and A as in (2), we similarly prove: every K -trivial real is $\leq_T Z$.

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Thank you for your attention!