Open Problems in Reverse Mathematics

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The basic reference for reverse mathematics is my recently published book *Subsystems of Second Order Arithmetic* [32]. The web site for the book is www.math.psu.edu/simpson/sosoa/. This article is a write-up of some representative open problems in reverse mathematics. It was originally a handout for my 45-minute invited talk at the AMS-INS-SIAM conference *Computability Theory and Applications*, June 13–17, 1999, Boulder, Colorado.

Contents

1 Real Analysis and Topology **1**

2 Banach Space Theory **4**

3 Ramsey Theory **4**

4 WQO Theory **6**

5 Replacing RCA_0 by a Weaker Base Theory **8**

References **8**

1 Real Analysis and Topology

Much is known concerning reverse mathematics for real analysis and the topology of complete separable metric spaces. Some of the inspiration for this comes from recursive analysis [25] and Bishop-style constructivism [2]. We shall not discuss those connections here, but see my book [32] for more information.

In Giusto/Simpson [11] we presented a rather thorough reverse mathematics discussion of various notions of closed set, and of various forms of the Tietze extension theorem for real-valued continuous functions on closed sets, in compact metric spaces. The purpose of this section is to call attention to one open problem left over from that paper.

Let X be a compact metric space. For concreteness we may take $X = [0, 1]$, the unit interval. In RCA_0 we define $K \subseteq X$ to be *closed* if it is the complement of a sequence of open balls; *separably closed* if it is the closure of a sequence of points; *located* if the distance function $d(x, K)$ exists as a continuous real-valued function on X; *weakly located* if the predicate $d(x, K) > r$ is Σ^0_1 (allowing parameters, of course). $\mathrm{C}(X)$ denotes the separable Banach space of continuous real-valued functions on X which have a modulus of uniform continuity. The *strong Tietze theorem* for K is the statement that every $\phi \in \mathrm{C}(K)$ extends to some $\widetilde{\phi} \in \mathrm{C}(X)$. See [11] for details.

Known results from [11] are:

(1) The strong Tietze theorem for closed, weakly located sets is provable in RCA_0.

(2) The strong Tietze theorem for separably closed sets is equivalent to WKL_0 over RCA_0.

There remain open questions concerning the status of

(3) the strong Tietze theorem for closed sets, and

(4) the strong Tietze theorem for closed, separably closed sets.

It is known from [11] that (3) and (4) are provable in WKL_0 and not provable in RCA_0. There is a partial reversal: (3) or (4) implies the DNR axiom over RCA_0. We shall outline the proof of this below. But first we discuss the DNR axiom.

The DNR axiom says: For every $A \subseteq \mathbb{N}$ there exists $f : \mathbb{N} \to \mathbb{N}$ which is *diagonally nonrecursive* relative to A, i.e., $f(n) \neq \{n\}^A(n)$ for all $n \in \mathbb{N}$. Here $\mathbb{N}$ is the set of natural numbers. It would be possible to restate the DNR axiom in a combinatorial way, not involving recursion theory, but we shall not do so here.

The DNR axiom is known to be weaker than WKL_0 ($= \mathsf{RCA}_0$ + weak König's lemma). Indeed, the DNR axiom is provable in the strictly weaker system WWKL_0 ($= \mathsf{RCA}_0 +$ weak weak König's lemma) which arises in connection with reverse mathematics for measure theory. (See [32, §X.1], [11], [4].) Because of Kumabe's result [21], it seems likely that the DNR axiom is strictly weaker than WWKL_0.

Recursion theorists can understand these variants of weak König's lemma in terms of separating sets, recursively bounded Π^0_1 classes, etc. Thus there is a close connection with Jockusch's talk at this conference. In descending order we have:

1. WKL_0 is just RCA_0 plus any of the following, relativized to arbitrary $A \subseteq \mathbb{N}$:

(a) for every infinite recursive tree $T \subseteq \{0,1\}^{<\mathbb{N}}$, there exists a path through T.

(b) for every disjoint pair of r.e. sets, there exists a separating set.

(c) there exists a $\{0,1\}$-valued DNR function, i.e., a function $f : \mathbb{N} \to \{0,1\}$ such that $f(n) \neq \{n\}(n)$ for all $n \in \mathbb{N}$.

2. WWKL_0 is just RCA_0 plus either of the following, relativized to arbitrary $A \subseteq \mathbb{N}$:

   (a) for every recursive tree $T \subseteq \{0,1\}^{<\mathbb{N}}$ such that

$$\lim_n \frac{|\{\sigma \in T : \mathrm{lh}(\sigma) = n\}|}{2^n} \neq 0$$

   there exists a path through T.

   (b) there exists a 1-random real (see Kučera [18, 19, 20]).

3. The DNR axiom is equivalent over RCA_0 to the following, relativized to arbitrary $A \subseteq \mathbb{N}$:

   (a) there exists a DNR function, i.e., a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \neq \{n\}(n)$ for all $n \in \mathbb{N}$.

Unfortunately, we don't know much about how to use the DNR axiom in mathematical arguments. Unlike WKL_0 and WWKL_0, the DNR axiom seems weak and therefore difficult to apply.

We shall now end this section with an outline of the proof that the strong Tietze theorem for closed, separably closed subsets of $[0,1]$ implies the DNR axiom.

We may as well assume that weak König's lemma fails. For each n let I_n be the closed interval $[1/2^{2n+1}, 1/2^{2n}]$. Since weak König's lemma fails, the Heine/Borel covering lemma fails, so let (a_{nk}, b_{nk}), $k \in \mathbb{N}$, be a covering of I_n by open intervals with no finite subcovering. We may assume that these coverings are disjoint from one other.

If $\{n\}(n)$ is defined, let s_n be the least s such that $\{n\}_s(n)$ is defined, and put

$$J_n = I_n \setminus \bigcup_{k=0}^{s_n} (a_{nk}, b_{nk}).$$

Let

$$K = \{0\} \cup \bigcup \{J_n : \{n\}(n) \text{ is defined}\}.$$

It can be shown that $K \subseteq [0,1]$ is closed, separably closed, and not weakly located.

Define a real-valued continuous function $\phi(x) = \pm x$ on K, as follows. First let $p_i(x)$, $i \in \mathbb{N}$, be a fixed, one-to-one, recursive enumeration of $\mathbb{Q}[x]$, the ring of polynomials with rational coefficients in one indeterminate, x. Using this, define

$\phi(0) = 0$ and, for $\{n\}(n) = i$ and $x \in J_n$, $\phi(x) = x$ if $|p_i(z) - z| \geq 1/2^{2n+2}$ for some $z \in J_n$, $\phi(x) = -x$ otherwise. It can be shown that $\phi \in \mathrm{C}(K)$.

By the strong Tietze theorem for K, let $\widetilde{\phi} \in \mathrm{C}([0,1])$ be an extension of ϕ from K to all of $[0,1]$. By Weierstrass polynomial approximation in RCA_0, let $p_{i_n}(x)$, $n \in \mathbb{N}$, be a sequence of polynomials such that $\sup\{|\widetilde{\phi}(x) - p_{i_n}(x)| : 0 \leq x \leq 1\} < 1/2^{2n+2}$ for all n. It is not difficult to show that the function $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = i_n$ is DNR.

By relativizing the above to an arbitrary $A \subseteq \mathbb{N}$, we get a function that is DNR relative to A. This completes the proof.

Note: Recursion theorists may want to view the above as a standard diagonal construction leading to a recursive counterexample to (4). However, from the viewpoint of reverse mathematics, there seems to be something unusual going on here. Usually, a recursive counterexample leads to a reversal to ACA_0 or WKL_0, but in this instance all we seem to get is a reversal to the DNR axiom.

2 Banach Space Theory

Regarding the reverse mathematics status of well known theorems of functional analysis, much is known, but many questions remain.

For example, the open mapping theorem for separable Banach spaces is known to be provable in a system called RCA_0^+ which is of the same strength as RCA_0 and indeed conservative over RCA_0 for Π^1_1 sentences. But whether it is provable in RCA_0 or even WKL_0 remains unknown. See Brown/Simpson [5], Mytilinaios/Slaman [24], Simpson [31].

As another example, consider the Krein/Šmulian theorem for separable Banach spaces. This is a somewhat lesser known but still basic theorem of functional analysis. It says that a convex set in the dual of a separable Banach space is weak-$*$-closed if and only if it is bounded-weak-$*$-closed. It is known from Humphreys/Simpson [16] that this statement is provable in ACA_0, but the exact strength is unknown.

3 Ramsey Theory

One indication of a bright future for reverse mathematics is that a number of outstanding recursion theorists are gradually getting drawn in. Of the researchers on Cholak's computability theory home page, several have published papers on reverse mathematics, including: Peter Cholak, Peter Clote, Rod Downey, Bill Gasarch, Jeff Hirst, Carl Jockusch, Steffen Lempp, Richard Shore, Ted Slaman. A recent major contribution is the Cholak/Jockusch/Slaman paper [7] on Ramsey's theorem for pairs.

Ramsey theory is a large subject with many interesting results in addition to the familiar Ramsey theorem. Moreover, there are important connections between Ramsey theory and dynamical systems theory, especially topological dynamics and ergodic theory. For a survey of the area, see the monograph of Graham/Rothschild/Spencer [12].

The purpose of this section is to mention some open problems regarding reverse mathematics and Ramsey theory. Such problems may be especially attractive, because Ramsey theory is a branch of mathematics where one could expect to find statements of high logical strength.

Hindman's Theorem

A famous and important Ramsey-type result is *Hindman's theorem*:

> For any coloring of $\mathbb{N}$ with finitely many colors, there exists an infinite set $H \subseteq \mathbb{N}$ such that all sums of finite subsets of H have the same color.

Hindman's theorem is well known to be closely related to the *Auslander/Ellis theorem* in topological dynamics:

> For every state x in a compact dynamical system, there exists a state y which is proximal to x and uniformly recurrent.

(A *compact dynamical system* consists of a compact metric space X and a continuous function $T : X \to X$. A state $x \in X$ is said to be *uniformly recurrent* if for all $\epsilon > 0$ there exists m such that for all n there exists $k < m$ such that $d(T^{n+k}x, x) < \epsilon$. Two states $x, y \in X$ are said to be *proximal* if for all $\epsilon > 0$ there exist infinitely many n such that $d(T^n x, T^n y) < \epsilon$.)

There has been a great deal of interest in the constructive or effective aspect of Hindman's theorem and the Auslander/Ellis theorem. Some of the known proofs are highly set-theoretical and cannot even be formalized in second-order arithmetic. For an extensive discussion, including several proofs of Hindman's theorem, see [12].

I conjecture that Hindman's theorem and the Auslander/Ellis theorem are equivalent to ACA_0 over RCA_0. The known partial results in this direction are in Blass/Hirst/Simpson [3]. There we showed that Hindman's theorem and the Auslander/Ellis theorem are provable in ACA_0^+, which consists of ACA_0 plus "for all $A \subseteq \mathbb{N}$, the ωth Turing jump $A^{(\omega)}$ of A exists". The proof of Hindman's theorem in ACA_0^+ involves a delicate effectivization of Hindman's original proof. We also obtained a reversal by showing that Hindman's theorem implies ACA_0 over RCA_0. The problem here is to close the gap between ACA_0 and ACA_0^+.

Szemerédi's Theorem

Another well known result of Ramsey theory is *Van der Waerden's theorem*:

> If $\mathbb{N}$ is colored with finitely many colors, then one of the colors contains arithmetic progressions of arbitrary finite length.

Using a method of Shelah, Van der Waerden's theorem is known to be provable in RCA_0. The so-called "density version" of Van der Waerden's theorem is due to Szemerédi:

> If $A \subseteq \mathbb{N}$ is such that
>
> $$\limsup_{n\to\infty} \frac{|A \cap \{1,\ldots,n\}|}{n} > 0$$
>
> then A contains arithmetic progressions of arbitrary finite length.

Essentially nothing is known about the strength of Szemerédi's theorem. In particular it is unknown whether Szemerédi's theorem is provable in ACA_0.

The Dual Ramsey Theorem

Let $(\mathbb{N})^k$ denote the set of partitions of $\mathbb{N}$ into exactly k nonempty pieces. Let $(\mathbb{N})^\infty$ denote the set of partitions of $\mathbb{N}$ into infinitely many nonempty pieces. For $X \in (\mathbb{N})^\infty$, $(X)^k$ is the set of all $Y \in (\mathbb{N})^k$ such that Y is coarser than X. The dual Ramsey theorem of Carlson/Simpson [6] reads as follows:

> If $(\mathbb{N})^k$ is colored with finitely many Borel colors, then there exists $X \in (\mathbb{N})^\infty$ such that $(X)^k$ is monochromatic.

This also holds with k replaced by ∞. There are some important generalizations of this due to Carlson. This kind of result has been used by Furstenberg and Katznelson to obtain a "density version" of the Hales/Jewett theorem. For references, see my book [32, remark X.3.6].

There are many open problems concerning the strength of the dual Ramsey theorem and related theorems. Slaman [34] has shown that the dual Ramsey theorem is provable in Π^1_1-CA_0. No interesting reversal is known.

Let A be a fixed finite alphabet. A^* denotes the set of *words*, i.e., finite strings of elements of A. An *infinite variable word* is an infinite string W of elements of $A \cup \{x_n : n \in \mathbb{N}\}$ such that each x_n occurs at least once, and all occurrences of x_n precede all occurrences of x_{n+1}, for all n. If $s = a_0 \cdots a_{n-1} \in A^*$, we denote by $W(s)$ the word which results from W upon replacing all occurrences of x_m by a_m for each $m < n$, then truncating just before the first occurrence of x_n. A key lemma of Carlson/Simpson [6] reads as follows:

> If A^* is colored with finitely many colors, then there exists an infinite variable word W such that $W(A^*) = \{W(s) : s \in A^*\}$ is monochromatic.

The strength of this lemma is unknown. In particular, it is unknown whether this lemma is true recursively, *i.e.*, whether W can be taken to be recursive in the given coloring. For more background on this problem, see Simpson [30].

4 WQO Theory

We now turn from Ramsey theory to another important branch of combinatorics: WQO theory. Like Ramsey theory, WQO theory is of special interest from the viewpoint of reverse mathematics, because many of the proofs seem to need unusually strong set existence axioms.

A *quasiordering* is a set Q together with a reflexive, transitive relation $\leq$ on Q. A *well quasiordering* (abbreviated WQO) is a quasiordering such that for every function $f : \mathbb{N} \to Q$ there exist $m, n \in \mathbb{N}$ such that $m < n$ and $f(m) \leq f(n)$. Let $[\mathbb{N}]^\infty$ be the space of infinite subsets of $\mathbb{N}$. A *better quasiordering* (abbreviated BQO) is a quasiordering such that for every Borel function $f : [\mathbb{N}]^\infty \to Q$ there exists $X \in [\mathbb{N}]^\infty$ such that $f(X) \leq f(X \setminus \{\min(X)\})$. It can be shown that every BQO is a WQO but not conversely.

Generally speaking, WQO theory is an appropriate tool when considering quasiorderings of finite structures, but BQO theory is better adapted to infinite structures. For example, a famous theorem of WQO theory is *Kruskal's theorem*:

> Finite trees are WQO under embeddability.

(Here a *tree* is a connected acyclic graph, and *embeddings* are required to take vertices to vertices, and edges to paths.) Kruskal's theorem has been generalized to infinite trees, but the proof is much more difficult and involves BQO theory. Detailed references are in [32, §X.3].

There are some important results about the strength of various theorems of WQO theory. For instance, Friedman (see Simpson [29]) showed that Kruskal's theorem is not provable in ATR_0, and he characterized exactly the strength of Kruskal's theorem, in proof-theoretic terms. This had remarkable consequences for Friedman's foundational program of finding mathematically natural, finite combinatorial statements which are proof-theoretically strong.

Consider now the following generalization of Kruskal's theorem, due to Kriz [17]. Let T_1 and T_2 be finite trees where each edge is labeled with a positive integer. Write $T_1 \leq T_2$ to mean that there exists an embedding of T_1 into T_2 such that the label of each edge of T_1 is less than or equal to the minimum of the labels of the corresponding edges of T_2. *Kriz's theorem* says that this quasiordering is a WQO.

What is the strength of Kriz's theorem? By results of Friedman (see Simpson [29]), Kriz's theorem is at least as strong as Π^1_1-CA_0. It may be much stronger, but little is known. This is an open problem which may have a big payoff.

We now consider a famous theorem of BQO theory. If Q is a countable quasiordering, let $\widetilde{Q}$ be the set of countable transfinite sequences of elements of Q. Quasiorder $\widetilde{Q}$ by putting $s \leq t$ if and only if there exists a one-to-one order-preserving map $f : \mathrm{lh}(s) \to \mathrm{lh}(t)$ such that $s(i) \leq t(f(i))$ for all $i < \mathrm{lh}(s)$. The *Nash-Williams transfinite sequence theorem* [28, 35] says that if Q is BQO then $\widetilde{Q}$ is BQO.

Marcone [23] has shown that the Nash-Williams theorem is provable in Π^1_1-CA_0 but not equivalent to Π^1_1-CA_0. Shore [26] has shown that the Nash-Williams theorem implies ATR_0 over RCA_0. There remains the problem of closing the gap. We conjecture that the Nash-Williams theorem is provable in ATR_0, hence equivalent to ATR_0 over RCA_0.

Another famous theorem of BQO theory is *Laver's theorem* [28, 35]:

The set of all countable linear orderings is WQO (in fact BQO) under embeddability.

The strength of Laver's theorem is an open problem. Shore [26] has shown that Laver's theorem implies ATR_0 over RCA_0, and we conjecture that Laver's theorem is provable in ATR_0.

5 Replacing RCA_0 by a Weaker Base Theory

In all but section X.4 of my book [32], RCA_0 is taken as the base theory for reverse mathematics. That is to say, reversals are stated as theorems of RCA_0. An important research direction for the future is to replace RCA_0 by weaker base theories. In this way we can hope to substantially broaden the scope of reverse mathematics, by obtaining reversals for many ordinary mathematical theorems which are provable in RCA_0.

A start on this has already been made. In Simpson/Smith [33] we defined RCA_0^* to be the same as RCA_0 except that Σ^0_1 induction is weakened to Σ^0_0 induction, and exponentiation of natural numbers is assumed. Thus RCA_0 is equivalent to RCA_0^* plus Σ^0_1 induction. It turns out that RCA_0^* is conservative over EFA (elementary function arithmetic) for Π^0_2 sentences, just as RCA_0 is conservative over PRA (primitive recursive arithmetic) for Π^0_2 sentences.

One project for the future is to redo all of the known results in reverse mathematics using RCA_0^* as the base theory. The groundwork for this has already been laid, but there are some difficulties. For example, we know that Ramsey's theorem for exponent 3 is equivalent to ACA_0 over RCA_0, but it unclear whether RCA_0 can be replaced by RCA_0^*. Other problems of this nature are listed in my book [32, remark X.4.3].

Another project is to find ordinary mathematical theorems that are equivalent to Σ^0_1 induction over RCA_0^*. Several results of this kind are already known and are mentioned in my book [32, §X.4]. For example, Hatzikiriakou [14] has shown that the well known structure theorem for finitely generated Abelian groups is equivalent to Σ^0_1 induction over RCA_0^*.

A more visionary project would be to replace RCA_0^* by even weaker base theories, dropping exponentiation and Δ^0_1 comprehension. One could even consider base theories that are conservative over the theory of discrete ordered rings. At the present time, almost nothing is known about this.

References

[1] K. Ambos-Spies, G. H. Müller, and G. E. Sacks, editors. *Recursion Theory Week*, number 1432 in Lecture Notes in Mathematics. Springer-Verlag, 1990. IX + 393 pages.

[2] Errett Bishop and Douglas Bridges. *Constructive Analysis*. Number 279 in Grundlehren der mathematischen Wißenschaften. Springer-Verlag, 1985. XII + 477 pages.

[3] Andreas R. Blass, Jeffry L. Hirst, and Stephen G. Simpson. Logical analysis of some theorems of combinatorics and topological dynamics. In [27], pages 125–156, 1987.

[4] Douglas K. Brown, Mariagnese Giusto, and Stephen G. Simpson. Vitali's theorem and WWKL. *Archive for Mathematical Logic*, 1999. 18 pages, accepted April 1998, to appear.

[5] Douglas K. Brown and Stephen G. Simpson. The Baire category theorem in weak subsystems of second order arithmetic. *Journal of Symbolic Logic*, 58:557–578, 1993.

[6] Timothy J. Carlson and Stephen G. Simpson. A dual form of Ramsey's theorem. *Advances in Mathematics*, 53:265–290, 1984.

[7] Peter Cholak, Carl G. Jockusch, Jr., and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. *Journal of Symbolic Logic*, 1999. 71 pages, to appear.

[8] S. B. Cooper, T. A. Slaman, and S. S. Wainer, editors. *Computability, Enumerability, Unsolvability: Directions in Recursion Theory*, number 224 in London Mathematical Society Lecture Note Series. Cambridge University Press, 1996. VII + 347 pages.

[9] J. N. Crossley, J. B. Remmel, R. A. Shore, and M. E. Sweedler, editors. *Logical Methods*. Birkhäuser, 1993. 813 pages.

[10] H.-D. Ebbinghaus, G.H. Müller, and G.E. Sacks, editors. *Recursion Theory Week*, number 1141 in Lecture Notes in Mathematics. Springer-Verlag, 1985. IX + 418 pages.

[11] Mariagnese Giusto and Stephen G. Simpson. Located sets and reverse mathematics. *Journal of Symbolic Logic*, 1999. 37 pages, accepted August 1998, to appear.

[12] Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer. *Ramsey Theory*. Wiley, New York, 2nd edition, 1990. XI + 196 pages.

[13] L. A. Harrington, M. Morley, A. Scedrov, and S. G. Simpson, editors. *Harvey Friedman's Research on the Foundations of Mathematics*, Studies in Logic and the Foundations of Mathematics. North-Holland, 1985. XVI + 408 pages.

[14] Kostas Hatzikiriakou. Algebraic disguises of Σ^0_1 induction. *Archive for Mathematical Logic*, 29:47–51, 1989.

[15] W. Hodges, M. Hyland, C. Steinhorn, and J. Truss, editors. *Logic: From Foundations to Applications, Keele 1993*, Oxford Science Publications. Oxford University Press, 1996. XIII + 536 pages.

[16] A. James Humphreys and Stephen G. Simpson. Separable Banach space theory needs strong set existence axioms. *Transactions of the American Mathematical Society*, 348:4231–4255, 1996.

[17] Igor Kriz. Well-quasiordering finite trees with gap-condition. *Annals of Mathematics*, 130:215–226, 1989.

[18] Antonín Kučera. Measure, Π^0_1 classes and complete extensions of PA. In [10], pages 245–259, 1985.

[19] Antonín Kučera. Randomness and generalizations of fixed point free functions. In [1], pages 245–254, 1990.

[20] Antonín Kučera. On relative randomness. *Annals of Pure and Applied Logic*, 63:61–67, 1993.

[21] Masahiro Kumabe. A fixed point free minimal degree. Preprint, 48 pages, 1997.

[22] Richard Mansfield and Galen Weitkamp. *Recursive Aspects of Descriptive Set Theory*. Oxford Logic Guides. Oxford University Press, 1985. VI + 144 pages.

[23] Alberto Marcone. On the logical strength of Nash-Williams' theorem on transfinite sequences. In [15], pages 327–351, 1996.

[24] Michael E. Mytilinaios and Theodore A. Slaman. On a question of Brown and Simpson. In [8], pages 205–218, 1996.

[25] Marian B. Pour-El and J. Ian Richards. *Computability in Analysis and Physics*. Perspectives in Mathematical Logic. Springer-Verlag, 1988. XI + 206 pages.

[26] Richard A. Shore. On the strength of Fraïssé's conjecture. In [9], pages 782–813, 1993.

[27] S. G. Simpson, editor. *Logic and Combinatorics*, Contemporary Mathematics. American Mathematical Society, 1987. XI + 394 pages.

[28] Stephen G. Simpson. BQO theory and Fraïssé's conjecture. In [22], pages 124–138, 1985.

[29] Stephen G. Simpson. Nonprovability of certain combinatorial properties of finite trees. In [13], pages 87–117, 1985.

[30] Stephen G. Simpson. Recursion-theoretic aspects of the dual Ramsey theorem. In [10], pages 356–371, 1986.

[31] Stephen G. Simpson. Partial realizations of Hilbert's program. *Journal of Symbolic Logic*, 53:349–363, 1988.

[32] Stephen G. Simpson. *Subsystems of Second Order Arithmetic*. Perspectives in Mathematical Logic. Springer-Verlag, 1999. XIV + 445 pages.

[33] Stephen G. Simpson and Rick L. Smith. Factorization of polynomials and Σ^0_1 induction. *Annals of Pure and Applied Logic*, 31:289–306, 1986.

[34] Theodore A. Slaman. A note on the dual Ramsey theorem. 4 pages, unpublished, January 1997.

[35] Fons van Engelen, Arnold W. Miller, and John Steel. Rigid Borel sets and better quasi-order theory. In [27], pages 199–222, 1987.