Implicit Definability in Arithmetic

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Definitions.

Let $\mathbb{N}, +, \times, =$ be the natural number system.

For $X \subseteq \mathbb{N}$ we say that <u>X is arithmetical</u>
if X is first-order definable over $\mathbb{N}, +, \times, =$,
i.e., $X = \{n \in \mathbb{N} \mid (\mathbb{N}, +, \times, =) \models \Phi(n)\}$
for some first-order formula Φ.

For $X, Y \subseteq \mathbb{N}$ we say that <u>X is arithmetical in Y</u>
if X is first-order definable over $\mathbb{N}, +, \times, =, Y$.
We say that X, Y are <u>arithmetically equivalent</u>
(<u>arithmetically incomparable</u>) if each (neither)
is arithmetical in the other.

Let Pow($\mathbb{N}$) $= \{X \mid X \subseteq \mathbb{N}\}$ = the powerset of $\mathbb{N}$.
Let S be a subset of Pow($\mathbb{N}$).
We say that <u>S is arithmetical</u>
if $S = \{X \subseteq \mathbb{N} \mid (\mathbb{N}, +, \times, =, X) \models \Phi\}$
for some first-order sentence Φ.

For $X \subseteq \mathbb{N}$ we say that X is <u>implicitly arithmetical</u>
or <u>an arithmetical singleton</u>
if the singleton set $\{X\}$ is arithmetical.

Five Theorems.

1. There exists an arithmetical singleton which is not arithmetical.

2. There exist $X, Y \subseteq \mathbb{N}$ such that neither X nor Y is an arithmetical singleton, but $X \oplus Y = \{2n \mid n \in X\} \cup \{2n+1 \mid n \in Y\}$ is an arithmetical singleton.

3. If $S \subseteq \text{Pow}(\mathbb{N})$ is nonempty countable arithmetical, then some $X \in S$ is an arithmetical singleton.

4. There are arithmetically incomparable $X, Y \subseteq \mathbb{N}$ such that X and Y are arithmetical singletons.

5. There is a countable arithmetical $S \subseteq \text{Pow}(\mathbb{N})$ such that not all $X \in S$ are arithmetical singletons.

Theorem 3 is due to Hisao Tanaka, 1972, Proceedings of the American Mathematical Society.

Theorems 4 and 5 are due to Leo Harrington, 1975–1976, unpublished.

Some literature.

Hisao Tanaka, A property of arithmetic sets, Proceedings of the American Mathematical Society, 1972, 31, pages 521–524.

Leo Harrington, Arithmetically incomparable arithmetic singletons, April 1975, 28 pages, handwritten.

Leo Harrington, McLaughlin's conjecture, September 1976, 11 pages, handwritten.

Peter M. Gerdes, Harrington's solution to McLaughlin's conjecture and non-uniform self-moduli, December 2010, 27 pages, arXiv:1012.3427v1, submitted for publication.

Stephen G. Simpson, Implicit definability in arithmetic, April 2013, 14 pages, preprint, submitted for publication.

Proof of Theorem 1.

Let $T \subseteq \mathbb{N}$ be the truth set for arithmetic, i.e., the set of Gödel numbers of first-order sentences which are true in $\mathbb{N}, +, \times, =$. Tarski's Theorem on undefinability of truth says that T is not arithmetical. However, T is implicitly arithmetical, namely

$\forall \Phi \, \forall \Psi \, (\#(\Phi \land \Psi) \in T \iff (\#(\Phi) \in T \land \#(\Psi) \in T))$,

$\forall \Phi \, (\#(\exists n \, \Phi(n)) \in T \iff \exists n \, (\#(\Phi(n)) \in T))$, etc.,

where $\#(\Phi) =$ the Gödel number of Φ.

Proof of Theorem 2.

We use Cohen forcing over $\mathbb{N}, +, \times, =$.

Let $\{0,1\}^*$ be the set of bitstrings, i.e., finite sequences of 0's and 1's. For $\sigma, \tau \in \{0,1\}^*$ we write $\sigma^\frown\tau =$ the concatenation, σ followed by τ. We write $\sigma \subseteq \tau$ if σ is an initial segment of τ, i.e., $\sigma^\frown\rho = \tau$ for some $\rho \in \{0,1\}^*$.

A set $D \subseteq \{0,1\}^*$ is said to be dense if $\forall \sigma \, \exists \tau \, (\sigma \subseteq \tau$ and $\tau \in D)$.

Let D_n, $n \in \mathbb{N}$, be an enumeration of the dense sets which are arithmetical. We choose our enumeration to be arithmetical in T, the Tarski truth set.

Proof of Theorem 2, continued.

Define sequences of bitstrings
$\sigma_0 \subseteq \sigma_1 \subseteq \sigma_2 \subseteq \cdots \sigma_i \subseteq \sigma_{i+1} \subseteq \cdots$ and
$\tau_0 \subseteq \tau_1 \subseteq \tau_2 \subseteq \cdots \tau_i \subseteq \tau_{i+1} \subseteq \cdots$ as follows.

Stage 0. Let $\sigma_0 = \tau_0 = \langle\rangle =$ the empty sequence.

Stage $3n+1$. Let $\sigma_{3n+1} = \sigma_{3n}{}^\frown\rho$ and $\tau_{3n+1} = \tau_{3n}{}^\frown\rho$ where ρ is the least member of $\{0,1\}^*$ such that $\sigma_{3n}{}^\frown\rho \in D_n$.

Stage $3n+2$. Let $\sigma_{3n+2} = \sigma_{3n+1}{}^\frown\rho$ and $\tau_{3n+2} = \tau_{3n+1}{}^\frown\rho$ where ρ is the least member of $\{0,1\}^*$ such that $\tau_{3n+1}{}^\frown\rho \in D_n$.

Stage $3n+3$. If $n \in T$ let $\sigma_{3n+3} = \sigma_{3n+2}{}^\frown\langle 1\rangle$ and $\tau_{3n+3} = \tau_{3n+2}{}^\frown\langle 0\rangle$. If $n \notin T$ let $\sigma_{3n+3} = \sigma_{3n+2}{}^\frown\langle 0\rangle$ and $\tau_{3n+3} = \tau_{3n+2}{}^\frown\langle 1\rangle$.

Let $X, Y \subseteq \mathbb{N}$ be such that $\bigcup_i \sigma_i$ and $\bigcup_i \tau_i$ are the characteristic functions of X and Y respectively.

Clearly X and Y are Cohen generic over $\mathbb{N}, +, \times, =$. Hence X and Y are not arithmetical singletons. However, $X \oplus Y$ is arithmetically equivalent to T, hence $X \oplus Y$ is an arithmetical singleton.

Recursion-theoretic concepts.

In order to prove Theorems 3, 4, 5 we use recursion-theoretic concepts and notation.

From now on, instead of working with subsets of $\mathbb{N}$, we shall work with functions from $\mathbb{N}$ into $\mathbb{N}$.

Let $\mathbb{N}^{\mathbb{N}} = \{X \mid X : \mathbb{N} \to \mathbb{N}\}$ = the Baire space. Within $\mathbb{N}^{\mathbb{N}}$ we consider points $X, Y, \ldots \in \mathbb{N}^{\mathbb{N}}$ and sets $P, Q, \ldots \subseteq \mathbb{N}^{\mathbb{N}}$.

For $e, i, j \in \mathbb{N}$ and $X \in \mathbb{N}^{\mathbb{N}}$ we write $\{e\}^X(i) \simeq j$ to mean: the program with Gödel number e using oracle X and input i eventually halts with output j.

We write $\{e\}^X(i)\downarrow$ to mean that $\exists j\,(\{e\}^X(i) \simeq j)$.

We write $X \leq_{\mathrm{T}} Y$ to mean that X is Turing reducible to Y, i.e., $\exists e\,\forall i\,(X(i) = \{e\}^Y(i))$.

The Turing jump of $X \in \mathbb{N}^{\mathbb{N}}$ is $X' \in \mathbb{N}^{\mathbb{N}}$ defined by $X'(e) = 1$ if $\{e\}^X(e)\downarrow$, otherwise $X'(e) = 0$.

The iterated Turing jumps $X^{(n)}$, $n \in \mathbb{N}$, are defined inductively by $X^{(0)} = X$ and $X^{(n+1)} = (X^{(n)})'$.

Fact: X is arithmetical in $Y \iff \exists n\,(X \leq_{\mathrm{T}} Y^{(n)})$.

Recursion-theoretic concepts, continued.

For $X, Y \in \mathbb{N}^{\mathbb{N}}$ we define $X \oplus Y \in \mathbb{N}^{\mathbb{N}}$ by

$(X \oplus Y)(2i) = X(i)$ and $(X \oplus Y)(2i+1) = Y(i)$.

A predicate $R \subseteq (\mathbb{N}^{\mathbb{N}})^k \times \mathbb{N}^n$ is said to be <u>computable</u> if $\exists e \, \forall X_1 \cdots \forall X_k \, \forall i_1 \cdots \forall i_n \, (\{e\}^{X_1 \oplus \cdots \oplus X_k}(i_1, \ldots, i_n) = 1$ if $R(X_1, \ldots, X_k, i_1, \ldots, i_n)$, $\{e\}^{X_1 \oplus \cdots \oplus X_k}(i_1, \ldots, i_n) = 0$ if $\neg R(X_1, \ldots, X_k, i_1, \ldots, i_n))$.

A set $P \subseteq \mathbb{N}^{\mathbb{N}}$ is said to be $\Pi^{0,Y}_n$ if and only if

$$P = \{X \mid \forall i_1 \, \exists i_2 \, \forall i_3 \cdots i_n \, R(X, Y, i_1, \ldots, i_n)\}$$

for some computable predicate $R \subseteq (\mathbb{N}^{\mathbb{N}})^2 \times \mathbb{N}^n$. We also write $\Pi^0_n = \Pi^{0,0}_n$ where $0 \in \mathbb{N}^{\mathbb{N}}$ is defined by $0(i) = 0$ for all i.

Note: There are n alternating quantifiers. The last quantifier is $\forall i_n$ if n is odd, $\exists i_n$ if n is even.

<u>Fact</u>: P is arithmetical $\Longleftrightarrow \exists n \, (P$ is $\Pi^0_n)$.

<u>Caution</u>: P is arithmetical if $\exists n \, (P$ is $\Pi^{0,0^{(n)}}_1)$. However, for sets $P \subseteq \mathbb{N}^{\mathbb{N}}$, the converse may fail.

<u>Fact</u>: P is topologically closed $\Longleftrightarrow \exists Y \, (P$ is $\Pi^{0,Y}_1)$.

Proof of Theorem 3.

Suppose $P \subseteq \mathbb{N}^{\mathbb{N}}$ is arithmetical.
For example, suppose P is Π^0_3, say

$$P = \{X \mid \forall i \, \exists j \, \forall n \, R(X,i,j,n)\}$$

where R is computable.

Given X, we replace the quantifiers $\exists j$ and $\forall n$ by canonical Skolem functions f and g.

Let $g(i,j) = 1 +$ the least n such that $\neg R(X,i,j,n)$ if such an n exists, otherwise $g(i,j) = 0$.

Note: $g(i,j) = 0 \equiv \forall n \, R(X,i,j,n)$.

Let $f(i) = 1 +$ the least j such that $g(i,j) = 0$ if such a j exists, otherwise $f(i) = 0$.

Note: $f(i) > 0 \equiv \exists j \, (g(i,j) = 0) \equiv \exists j \, \forall n \, R(X,i,j,n)$.
Hence $X \in P \equiv \forall i \, (f(i) > 0)$.

Note: $X \oplus f \oplus g$ is arithmetically equivalent to X.

Define $F : P \to \mathbb{N}^{\mathbb{N}}$ by $F(X) = X \oplus f \oplus g$.
Clearly F is a one-to-one correspondence between P and $Q = \{F(X) \mid X \in P\}$.
Clearly Q is Π^0_1, hence topologically closed.

Assume P is countable and nonempty. Then Q is countable and nonempty, hence there exists $Y \in Q$ such that Y is isolated in Q. It follows that Y is a Π^0_1 singleton. Let $X \in P$ be such that $F(X) = Y$. Then X is an arithmetical singleton, Q.E.D.

Proof of Theorems 4 and 5.

To prove Theorems 4 and 5 we use treemaps.

Let $\mathbb{N}^* =$ the set of strings, i.e., finite sequences of natural numbers. For $\sigma, \tau \in \mathbb{N}^*$ we write $\sigma \subseteq \tau$ if σ is an initial segment of τ, i.e., $\sigma^\frown\rho = \tau$ for some $\rho \in \mathbb{N}^*$. For $X \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$ we write

$$X{\restriction}n = \langle X(0), X(1), \ldots, X(n-1)\rangle$$

and note that $X{\restriction}n \in \mathbb{N}^*$.

A tree is a set $T \subseteq \mathbb{N}^*$ such that

$$\forall\sigma\,\forall\tau\,(\sigma \subseteq \tau, \tau \in T \Rightarrow \sigma \in T).$$

In this case we write

$[T] = \{X \in \mathbb{N}^\mathbb{N} \mid \forall n\,(X{\restriction}n \in T)\} = \{$paths through $T\}$.

Fact: A set $P \subseteq \mathbb{N}^\mathbb{N}$ is $\Pi^{0,A}_1$

$\iff P = [T]$ where T is $\Pi^{0,A}_1$,

$\iff P = [T]$ where T is A-recursive.

A treemap is a function $F : T \to \mathbb{N}^*$ such that

$$\forall\sigma\,\forall i\,(\sigma^\frown\langle i\rangle \in T \Rightarrow F(\sigma)^\frown\langle i\rangle \subseteq F(\sigma^\frown\langle i\rangle)).$$

In this case we have another tree

$$F(T) = \{\tau \mid \exists\sigma\,(\tau \subseteq F(\sigma),\, \sigma \in T)\}$$

and F induces a homeomorphism $F : [T] \cong [F(T)]$ given by $F(X) = \bigcup\{F(X{\restriction}n) \mid n \in \mathbb{N}\}$ for all $X \in [T]$.

A treemap $F : T \to \mathbb{N}^*$ is said to be A-recursive if $\exists e\,\forall\sigma\,(\sigma \in T \Rightarrow F(\sigma) = \{e\}^A(\sigma))$. It is easy to see that if F and T are A-recursive then $F(T)$ is A-recursive.

Lemma 1. Given a $\Pi^{0,A'}_1$ set P we can find a $\Pi^{0,A}_1$ set Q and an A-recursive treemap $F : P \cong Q$. Thus $X \oplus A \equiv_T F(X) \oplus A$ uniformly for all $X \in P$.

Proof. P is $\Pi^{0,A}_2$, say $P = \{X \mid \forall i\, \exists j\, R(X, i, j)\}$ where R is A-recursive. Let $F(X) = X \oplus f$ where $f(i) =$ the least j such that $R(X, i, j)$ holds. Clearly F is A-recursive, and it can be shown that F is a treemap.

Lemma 2. Given a $\Pi^{0,A'}_1$ set P we can find a $\Pi^{0,A}_1$ set Q and an A'-recursive treemap $H : P \cong Q$ such that $X \oplus A' \equiv_T H(X) \oplus A' \equiv_T (H(X) \oplus A)'$ uniformly for all $X \in P$.

Proof. We first construct a particular A'-recursive treemap $G : \mathbb{N}^* \to \mathbb{N}^*$. Begin with $G(\langle\rangle) = \langle\rangle$. If $G(\sigma)$ has been defined, let $e = |\sigma| =$ the length of σ, and for each i let $G(\sigma^\frown\langle i\rangle) =$ the least $\tau \supseteq G(\sigma)^\frown\langle i\rangle$ such that $\{e\}^{\tau\oplus A}_{|\tau|}(e)\downarrow$ if such a τ exists, otherwise $G(\sigma^\frown\langle i\rangle) = G(\sigma)^\frown\langle i\rangle$. Note that for all X we have $X \oplus A' \equiv_T G(X) \oplus A' \equiv_T (G(X) \oplus A)'$ uniformly.

Since $G(P)$ is $\Pi^{0,A'}_1$, apply Lemma 1 to get an A-recursive treemap $F : G(P) \cong Q$ where Q is $\Pi^{0,A}_1$. Then $H = F \circ G : P \cong Q$ is our desired H.

Lemma 3. Given a $\Pi_1^{0,0^{(n)}}$ set P_n we can find a Π_1^0 set P_0 and a $0^{(n)}$-recursive treemap $H : P_n \cong P_0$ such that $X \oplus 0^{(n)} \equiv_T H(X) \oplus 0^{(n)} \equiv_T H(X)^{(n)}$ uniformly for all $X \in P_n$.

Proof. Apply Lemma 2 n times. We then have $P_n \cong \cdots \cong P_i \cong P_{i-1} \cong \cdots \cong P_0$ and P_i is $\Pi_1^{0,0^{(i)}}$ and for each $i = 1, \ldots, n$ we have a treemap $H_i : P_i \cong P_{i-1}$ such that, uniformly for all $X \in P_i$, $X \oplus 0^{(i)} \equiv_T H_i(X) \oplus 0^{(i)} \equiv_T (H_i(X) \oplus 0^{(i-1)})'$. Our desired $H : P_n \cong P_0$ is $H_1 \circ \cdots \circ H_n$.

We now prove weak forms of Theorems 4 and 5.

Theorem 4* (simplified version of Theorem 4). For each n there are Π_1^0 singletons X, Y such that $X \not\leq_T Y^{(n)}$ and $Y \not\leq_T X^{(n)}$.

Proof. Let X_n and Y_n be such that $0^{(n)} \leq_T X_n \leq_T 0^{(n+1)}$ and $0^{(n)} \leq_T Y_n \leq_T 0^{(n+1)}$ and $X_n \not\leq_T Y_n$ and $Y_n \not\leq_T X_n$. Then X_n and Y_n are $\Pi_2^{0,0^{(n)}}$ singletons. We may safely assume that they are $\Pi_1^{0,0^{(n)}}$ singletons. Let $P_n = \{X_n, Y_n\}$. Apply Lemma 3 to get $H : P_n \cong P_0$. Let $X_0 = H(X_n)$ and $Y_0 = H(Y_n)$. Then X_0, Y_0 are the desired X, Y.

Theorem 5* (simplified version of Theorem 5). For each n there is a countable Π^0_1 set P such that some $Z \in P$ is not a Π^0_n singleton.

Proof. Let P_n be a countable Π^0_1 set such that some $Z_n \in P_n$ is not isolated in P_n. Apply Lemma 3 to get $H : P_n \cong P_0$. Let $Z_0 = H(Z_n)$. Then P_0 and Z_0 are the desired P and Z.

Details: H is a homeomorphism of P_n onto P_0, so Z_0 is not isolated in P_0. Suppose Z_0 were a Π^0_n singleton, say $\{Z_0\} = \{X_0 \mid X_0^{(n)}(e) = 0\}$. Since $X_n \oplus 0^{(n)} \equiv_T H(X_n)^{(n)}$ uniformly for $X_n \in P_n$, let j be such that $H(X_n)^{(n)}(e) = 0$ for all $X_n \in P_n$ such that $X_n \restriction j = Z_n \restriction j$. Since Z_n is not isolated in P_n, let $X_n \in P_n$ be such that $X_n \restriction j = Z_n \restriction j$ and $X_n \neq Z_n$. Letting $X_0 = H(X_n)$ we have $X_0^{(n)}(e) = 0$ and $X_0 \neq Z_0$. This is a contradiction.

We now turn to the proofs of Theorems 4 and 5.

To prove Theorems 4 and 5, we extend Lemma 3 replacing n by ω, the first infinite ordinal number. For $X \in \mathbb{N}^{\mathbb{N}}$ define the $\underline{\omega\text{-jump}}$ $X^{(\omega)} \in \mathbb{N}^{\mathbb{N}}$ by

$$X^{(\omega)} = X \oplus X' \oplus X'' \oplus \cdots \oplus X^{(n)} \oplus \cdots.$$

More precisely, let $X^{(\omega)}((n,i)) = X^{(n)}(i)$ where $(\cdot,\cdot)$ is a recursive one-to-one correspondence between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$.

Lemma 3* (Lemma 3 with n replaced by ω).

Given a $\Pi^{0,0^{(\omega)}}_1$ set P_ω we can find a Π^0_1 set P_0 and a $0^{(\omega)}$-recursive treemap $H : P_\omega \cong P_0$ such that $X \oplus 0^{(\omega)} \equiv_{\mathrm{T}} H(X) \oplus 0^{(\omega)} \equiv_{\mathrm{T}} H(X)^{(\omega)}$ uniformly for all $X \in P_\omega$.

In our proof of Lemma 3*, we shall exploit the uniformity of Lemma 2, as we now explain.

Let T^A_e, $e \in \mathbb{N}$, be a standard enumeration of all $\Pi^{0,A}_1$ trees. Then $P^A_e = [T^A_e]$, $e \in \mathbb{N}$, is a standard enumeration of all $\Pi^{0,A}_1$ sets.

<u>Lemma 2 holds uniformly.</u> Namely, we can find a recursive function h and, for all e and A, an A'-recursive treemap $H : P^{A'}_e \cong P^A_{h(e)}$ such that $X \oplus A' \equiv_{\mathrm{T}} H(X) \oplus A' \equiv_{\mathrm{T}} (H(X) \oplus A)'$ uniformly for all $X \in P^{A'}_e$.

Proof of Lemma 3*.

Let T_ω be a $0^{(\omega)}$-recursive tree such that $P_\omega = [T_\omega]$. We can choose T_ω in such a way that $\{\sigma \in T_\omega \mid |\sigma| \le n\} \le_\mathrm{T} 0^{(n)}$ uniformly for all n.

Let $T_{e,n}$ be a $\Pi_1^{0,0^{(n)}}$ tree consisting of $\{\sigma \mid |\sigma| \le n\}$ together with $\{\tau \mid |\tau| > n,\ \tau{\restriction}n \in T_\omega,\ \tau \in T_e^{\langle n\rangle^\frown 0^{(n)}}\}$. Note that $P_{e,n} = [T_{e,n}]$ is a $\Pi_1^{0,0^{(n)}}$ set.

Using the uniformity of Lemma 2, we can find a recursive function h^* and, for $n \ge 1$, $0^{(n)}$-recursive treemaps $H_{e,n} : P_{e,n} \cong P_{h^*(e),n-1}$ such that $X \oplus 0^{(n)} \equiv_\mathrm{T} H_{e,n}(X) \oplus 0^{(n)} \equiv_\mathrm{T} (H_{e,n}(X) \oplus 0^{(n-1)})'$ uniformly for all $X \in P_{e,n}$, and furthermore $H_{e,n}(\sigma) = \sigma$ for all σ such that $|\sigma| \le n$.

By the Recursion Theorem, let e be a fixed point of h^*. Thus $T_e^A = T_{h^*(e)}^A$ for all A, so in particular $T_{e,n} = T_{h^*(e),n}$ for all n. Using this e, define $P_n = P_{e,n}$ and $H_n = H_{e,n}$ for all n. Thus P_n is a $\Pi_1^{0,0^{(n)}}$ set and $H_n : P_n \cong P_{n-1}$ is a treemap. Define $H : P_\omega \cong P_0$ by $H(\sigma) = (H_1 \circ \cdots \circ H_n)(\sigma)$ where $n = |\sigma|$. Then P_0 and H are as desired.

Recall: we obtained Theorems 4* and 5* as easy consequences of Lemma 3.

We shall now obtain Theorems 4 and 5 as easy consequences of Lemma 3*.

The proofs are exactly the same.

Proof of Theorem 4. Let X_ω and Y_ω be such that $0^{(\omega)} \leq_T X_\omega \leq_T 0^{(\omega+1)}$ and $0^{(\omega)} \leq_T Y_\omega \leq_T 0^{(\omega+1)}$ and $X_\omega \nleq_T Y_\omega$ and $Y_\omega \nleq_T X_\omega$. Then X_ω and Y_ω are $\Pi_2^{0,0^{(\omega)}}$ singletons. We may safely assume that they are $\Pi_1^{0,0^{(\omega)}}$ singletons. Let $P_\omega = \{X_\omega, Y_\omega\}$. Apply Lemma 3* to get $H : P_\omega \cong P_0$. Let $X_0 = H(X_\omega)$ and $Y_0 = H(Y_\omega)$. Then X_0, Y_0 are the desired X, Y.

Proof of Theorem 5. Let P_ω be a countable Π_1^0 set such that some $Z_\omega \in P_\omega$ is not isolated in P_ω. Apply Lemma 3* to get $H : P_\omega \cong P_0$. Let $Z_0 = H(Z_\omega)$. Then P_0 and Z_0 are the desired P and Z.

THE END. THANK YOU!