A World Where Relative Definability Coincides With Relative Recursiveness (i.e., Turing Reducibility)

Stephen G. Simpson

Pennsylvania State University

http://www.math.psu.edu/simpson/

simpson@math.psu.edu

Philosophy of Mathematics Workshop

Carnegie-Mellon University

December 9–10, 2000

Outline of talk:

1. Reverse mathematics, SOSOA, FOM.

2. ω-models of WKL_0.

3. Forcing with Π^0_1 subsets of 2^ω.

4. Symmetric models of WKL_0.

5. Foundational significance.

6. Muchnik and Medvedev degrees of Π^0_1 subsets of 2^ω.

7. Hyperarithmetical analogs.

8. References.

Background:

Second order arithmetic is a two-sorted system.

Number variables $m, n, \ldots$ range over

$$\omega = \{0, 1, 2, \ldots\} \ .$$

Set variables $X, Y, \ldots$ range over subsets of ω.

We have $+$, $\times$, $=$ on ω, plus the membership relation

$$\in \quad = \quad \{(n, X) : n \in X\} \quad \subseteq \quad \omega \times P(\omega) \ .$$

Within subsystems of second order arithmetic, we can formalize rigorous mathematics (analysis, algebra, geometry, . . .).

Subsystems of second order arithmetic are the basis of our current understanding of the logical structure of contemporary mathematics.

Themes of Reverse Mathematics:

Let τ be a mathematical theorem. Let S_τ be the weakest natural subsystem of second order arithmetic in which τ is provable.

1. Very often, the principal axiom of S_τ is logically equivalent to τ.

2. Furthermore, only a few subsystems of second order arithmetic arise in this way.

For a full exposition, see my book.

Foundational consequences of Reverse Mathematics:

1. We demonstrate rigorously that certain particular subsystems of second order arithmetic are mathematically natural.

2. We precisely classify mathematical theorems, according to which subsystems they are provable in.

3. . . . .

4. . . . .

Book on Reverse Mathematics:

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Perspectives in Mathematical Logic
Springer-Verlag, 1999
XIV + 445 pages

Web: `www.math.psu.edu/simpson/sosoa/`

Order: 1-800-SPRINGER

List price: $60

Discount: 30 percent for ASL members, mention promotion code S206

The FOM mailing list:

FOM is an automated e-mail list for discussing foundations of mathematics. There are currently almost 500 subscribers. There have been more than 4700 postings.

FOM is maintained and moderated by S. Simpson. The FOM Editorial Board consists of M. Davis, H. Friedman, C. Jockusch, D. Marker, S. Simpson, A. Urquhart.

FOM postings and information are available on the web at

`www.math.psu.edu/simpson/fom/`

The purpose of FOM is to promote the idea that mathematical logic is or ought to be driven by f.o.m. considerations.

f.o.m. = foundations of mathematics.

The hierarchy of consistency strengths:

strong:
- supercompact cardinal
- $\vdots$
- measurable cardinal
- $\vdots$
- ZFC (ZF set theory with choice)
- Zermelo set theory
- simple type theory

medium:
- Z_2 (2nd order arithmetic)
- $\vdots$
- Π^1_2 comprehension
- Π^1_1 comprehension
- ATR_0 (arith. transfinite recursion)
- ACA_0 (arithmetical comprehension)

weak:
- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- EFA (elementary arithmetic)
- bounded arithmetic
- $\vdots$

An important system:

One of the most important subsystems of second order arithmetic is WKL_0.

WKL_0 includes Δ^0_1 comprehension (i.e., recursive comprehension) and Weak König's Lemma: every infinite subtree of the full binary tree has an infinite path.

Remarks on ω-models of WKL_0:

1. The ω-model

$$\mathrm{REC} = \{X : X \text{ is recursive}\}$$

is not an ω-model of WKL_0. (Kleene)

2. However, REC is the intersection of all ω-models of WKL_0. (Kreisel, "hard core")

Remarks on ω-models of WKL_0 (continued):

3. The ω-models of WKL_0 are just the *Scott systems*, i.e., $M \subseteq P(\omega)$ such that

(a) $M \neq \emptyset$.

(b) $X, Y \in M$ implies $X \oplus Y \in M$.

(c) $X \in M$, $Y \leq_T X$ imply $Y \in M$.

(d) If $T \in M$ is an infinite subtree of $2^{<\omega}$, then there exists $X \in M$ such that X is a path through T.

Dana Scott, Algebras of sets binumerable in complete extensions of arithmetic, *Recursive Function Theory*, AMS, 1962, pages 117–121.

Remarks on ω-models of WKL_0 (continued):

4. There is a close relationship between

(a) ω-models of WKL_0, and

(b) Π^0_1 subsets of 2^ω.

The recursion-theoretic literature is extensive, with numerous articles by Jockusch, Kučera, and others. A recent survey is:

Douglas Cenzer and Jeffrey B. Remmel, Π^0_1 classes in mathematics, *Handbook of Recursive Mathematics*, North-Holland, 1998, pages 623–821.

Main results of this talk:

Let $\mathcal{P}$ be the nonempty Π^0_1 subsets of 2^ω, ordered by inclusion. Forcing with $\mathcal{P}$ is known as *Jockusch/Soare forcing*.

Lemma (Simpson 2000). Let X be J/S generic. Suppose $Y \leq_T X$. Then (i) Y is J/S generic, and (ii) X is J/S generic relative to Y.

Theorem (Simpson 2000). There is an ω-model M of WKL_0 with the following property: For all $X, Y \in M$, X is definable from Y in M if and only if X is Turing reducible to Y.

Proof. M is obtained by iterated J/S forcing. We have

$$M = \mathrm{REC}[X_1, X_2, \ldots, X_n, \ldots]$$

where, for all n, X_{n+1} is J/S generic over $\mathrm{REC}[X_1, \ldots, X_n]$.

Corollary (Friedman 1974, unpublished, by a different method). There is an ω-model M of WKL_0 with the following property: For all $X \in M$, X is definable in M if and only if X is recursive.

Note: Friedman's 1974 manuscript contains another result which contradicts my theorem above concerning relative definability. Friedman's proof of this other result is erroneous.

A Π^0_1 set of ω-models of WKL_0:

Theorem (Simpson 2000). There is a nonempty Π^0_1 subset of 2^ω, P, with the following properties:

1. For all $X \in P$, $\{(X)_n : n \in \omega\}$ is a countable ω-model of WKL_0, and every countable ω-model of WKL_0 occurs in this way.

2. For all nonempty Π^0_1 sets $P_1, P_2 \subseteq P$ we can find a recursive homeomorphism

$$\Phi : P_1 \cong P_2$$

such that for all $X \in P_1$ and $Y \in P_2$, if $\Phi(X) = Y$ then

$$\{(X)_n : n \in \omega\} = \{(Y)_n : n \in \omega\} \; .$$

The proof uses an idea of Pour-El/Kripke 1967.

Foundational significance:

Foundations of mathematics (*f.o.m.*) is the study of the most basic concepts and logical structure of mathematics, with an eye to the unity of human knowledge.

General background in f.o.m.: the van Heijenoort volume; Gödel's Collected Works; the Friedman volume.

Specific background: recursive mathematics, i.e., the development of mathematics in the computable world, REC $= \{X : X$ is recursive$\}$. See Aberth, Pour-El/Richards, ...

Foundational significance (continued):

Regrettably, the assumption "all real numbers are computable" conflicts with many basic theorems of real analysis. E.g., the maximum principle for continuous real-valued functions on $[0, 1]$.

On the other hand, many such theorems are provable in WKL_0. This is a by-product of Reverse Mathematics.

To strike a balance, we can work in an ω-model of WKL_0 where all *definable* real numbers are computable. Thus many non-constructive theorems hold, yet REC is the "definable core".

Foundational significance (continued):

More generally, consider the scheme

> (*) For all X and Y, if X is definable from Y then X is computable from Y

in the language of second order arithmetic.

Simpson 2000 shows that, for every countable model of WKL_0, there exists a countable model of $\mathsf{WKL}_0 + (*)$ with the same first order part.

Thus $\mathsf{WKL}_0 + (*)$ is conservative over WKL_0 for first-order arithmetical sentences.

Often in mathematics, under some assumptions on a real parameter X, there exists a unique real Y having some property stated in terms of X. In this situation, (*) implies that Y is Turing reducible to X.

Two new structures in recursion theory:

$\mathcal{P}_w$ ($\mathcal{P}_M$) consists of the Muchnik (Medvedev) degrees of nonempty Π^0_1 subsets of 2^ω, ordered by Muchnik (Medvedev) reducibility.

P is Muchnik reducible to Q ($P \leq_w Q$) if for all $Y \in Q$ there exists $X \in P$ such that $X \leq_T Y$.

P is Medvedev reducible to Q ($P \leq_M Q$) if there exists a recursive functional $\Phi : Q \to P$.

Results and problems:

$\mathcal{P}_w$ and $\mathcal{P}_M$ are countable distributive lattices with a top and bottom element, call them 1 and 0. In $\mathcal{P}_w$ and $\mathcal{P}_M$, it is trivial that $P, Q > 0$ implies $\inf(P, Q) > 0$, but we do not know whether $P, Q < 1$ implies $\sup(P, Q) < 1$. In $\mathcal{P}_w$, for every $P > 0$, every countable distributive lattice is lattice-embeddable below P. For $\mathcal{P}_M$ we have partial results in this direction.

This is joint work with my Ph. D. student Stephen Binns (2000).

An invidious comparison:

The study of $\mathcal{P}_w$ and $\mathcal{P}_M$, the Muchnik and Medvedev degrees of nonempty Π^0_1 subsets of 2^ω, is in some ways parallel to the study of $\mathcal{R}_T$, the Turing degrees of recursively enumerable subsets of ω.

Analogy:

$$\frac{\mathcal{P}_w}{\mathcal{R}_T} = \frac{\mathsf{WKL}_0}{\mathsf{ACA}_0}$$

As is well known, there are no specific examples of recursively enumerable Turing degrees $\neq 1, 0$. (See the FOM discussion with Soare, July 1999.) In this respect, $\mathcal{P}_w$ and $\mathcal{P}_M$ are **much better**.

For example, the set of Muchnik degrees of Π^0_1 subsets of 2^ω of positive Lebesgue measure contains a maximum degree, which is $\neq 1, 0$.

Hyperarithmetical analogs:

Theorem (Simpson 2000). There is a countable β-model M such that, for all $X, Y \in M$, X is definable from Y in M if and only if X is hyperarithmetical in Y.

In the language of second order arithmetic, consider the scheme

> (**) for all X, Y, if X is definable from Y, then X is hyperarithmetical in Y.

Theorem (Simpson 2000).

1. $\mathsf{ATR}_0 + (**)$ is conservative over ATR_0 for Σ^1_2 sentences.

2. Π^1_∞-$\mathsf{TI}_0 + (**)$ is conservative over Π^1_∞-TI_0 for Σ^1_2 sentences.

References:

Stephen G. Simpson, Π^0_1 sets and models of WKL_0, preprint, April 2000, 28 pages, to appear.

Stephen G. Simpson, A symmetric β-model, preprint, May 2000, 7 pages, to appear.

These and other papers are available at
http://www.math.psu.edu/simpson/papers/.

These transparencies are available at
http://www.math.psu.edu/simpson/talks/.