Degrees of unsolvability: a three-hour tutorial

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Degrees of unsolvability: a three-hour tutorial.

Hour 1. Turing degrees, Muchnik degrees, the Muchnik topos.

Hour 2. Examples of Turing degrees and Muchnik degrees.

Hour 3. Muchnik degrees of nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$.

Our notation for degree structures:

$\mathcal{D}_\mathrm{T}$ = **the upper semilattice of all Turing degrees.**

$\mathcal{D}_\mathrm{w}$ = **the lattice of all Muchnik degrees.**

$\mathcal{E}_\mathrm{T}$ = the upper semilattice of recursively enumerable Turing degrees.

$\mathcal{E}_\mathrm{w}$ = the lattice of Muchnik degrees of nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$.

$\mathcal{S}_\mathrm{w}$ = the lattice of Muchnik degrees of nonempty Π^0_1 subsets of $\mathbb{N}^{\mathbb{N}}$.

Motivation: a non-rigorous "calculus of problems."

Given a "problem" P, it is natural to hope for an "easy solution" of P.

If P is "unsolvable" (i.e., has no "easy solution"), it is natural to ask "how unsolvable" P is. We therefore seek to measure the "amount" or "degree" of "unsolvability" which is inherent in P.

Let us say that P is "reducible" to another "problem" Q if, given any "solution" of Q, we can use it to "easily" find a "solution" of P.

If P and Q are "reducible" to each other, we say that they have the same "degree of unsolvability."

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There are many ways to convert these non-rigorous ideas into rigorous ones. We focus on two closely related degree structures: the *Turing degrees*, $\mathcal{D}_\mathsf{T}$, and the *Muchnik degrees*, $\mathcal{D}_\mathsf{w}$.

A Turing degree measures the unsolvability of a *decision problem*.

A Muchnik degree measures the unsolvability of a *mass problem*.

The Muchnik degrees are the *completion* of the Turing degrees.

Turing degrees versus Muchnik degrees. A decision problem has only one solution. A mass problem may have many different solutions.

A *decision problem* is a real $X \in \mathbb{N}^{\mathbb{N}}$. Intuitively, X represents the problem of "finding" or "computing" X.
This problem has only one solution, namely, X.

For $X, Y \in \mathbb{N}^{\mathbb{N}}$ we say that X *is Turing reducible to* Y, abbreviated $X \leq_{\mathsf{T}} Y$, if X is computable using Y as a Turing oracle.

A *Turing degree* is an equivalence class of decision problems under mutual Turing reducibility. The Turing degree of X is denoted $\deg_{\mathsf{T}}(X)$. The partial ordering of all Turing degrees is denoted $\mathcal{D}_{\mathsf{T}}$.

A *mass problem* is a subset of $\mathbb{N}^{\mathbb{N}}$. Intuitively, $P \subseteq \mathbb{N}^{\mathbb{N}}$ represents the problem of "finding" or "computing" some member of P.
Thus any $X \in P$ is a solution of this problem.

For $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$ we say that P *is Muchnik reducible to* Q, abbreviated $P \leq_{\mathsf{w}} Q$, if $\forall Y\,(Y \in Q \Rightarrow \exists X\,(X \in P$ and $X \leq_{\mathsf{T}} Y))$. In other words, using any solution of Q as an oracle, we can compute some solution of P.

A *Muchnik degree* is an equivalence class of mass problems under mutual Muchnik reducibility. The Muchnik degree of P is denoted $\deg_{\mathsf{w}}(P)$. The partial ordering of all Muchnik degrees is denoted $\mathcal{D}_{\mathsf{w}}$.

Turing degrees versus Muchnik degrees (continued).

Recall $\mathcal{D}_{\mathsf{T}} =$ the partial ordering of all Turing degrees, and $\mathcal{D}_{\mathsf{w}} =$ the partial ordering of all Muchnik degrees.

Identifying $\deg_{\mathsf{T}}(X)$ with $\deg_{\mathsf{w}}(\{X\})$, we have an order-preserving embedding $\deg_{\mathsf{T}}(X) \mapsto \deg_{\mathsf{w}}(\{X\}) : \mathcal{D}_{\mathsf{T}} \hookrightarrow \mathcal{D}_{\mathsf{w}}$.

This induces an order-reversing one-to-one correspondence between Muchnik degrees and upwardly closed sets of Turing degrees. The upwardly closed set corresponding to $\mathbf{p} \in \mathcal{D}_{\mathsf{w}}$ is $\{\mathbf{a} \in \mathcal{D}_{\mathsf{T}} \mid \mathbf{p} \le \mathbf{a}\}$.

Thus we may identify $\mathcal{D}_{\mathsf{w}} = \widehat{\mathcal{D}_{\mathsf{T}}} =$ <u>the completion of</u> $\mathcal{D}_{\mathsf{T}}$.

In particular, $\mathcal{D}_{\mathsf{w}}$ is a <u>complete and completely distributive lattice</u>.

$\mathcal{D}_{\mathsf{T}}$ is not even a lattice. However, $\mathcal{D}_{\mathsf{T}}$ is an <u>upper semilattice</u>. Namely, for all $X, Y \in \mathbb{N}^{\mathbb{N}}$ the Turing degree $\deg_{\mathsf{T}}(X \oplus Y) = \sup(\mathbf{a}, \mathbf{b})$ is the *supremum* (= l.u.b.) of $\deg_{\mathsf{T}}(X) = \mathbf{a}$ and $\deg_{\mathsf{T}}(Y) = \mathbf{b}$.

Also, $\mathcal{D}_{\mathsf{T}}$ has a bottom element, namely $\mathbf{0} = \deg_{\mathsf{T}}(0)$.

Our embedding of $\mathcal{D}_{\mathsf{T}}$ into $\mathcal{D}_{\mathsf{w}}$ preserves these features.

The completion of a partial ordering.

Our identification of $\mathcal{D}_{\mathsf{w}}$ as the completion of $\mathcal{D}_{\mathsf{T}}$ is an instance of a general construction.

Let $\mathcal{K}$ be any *partial ordering*, i.e., partially ordered set.
Let $\widehat{\mathcal{K}}$ be the set of upwardly closed subsets of $\mathcal{K}$, partially ordered by reverse inclusion, i.e., $\mathcal{U} \leq \mathcal{V}$ if and only if $\mathcal{U} \supseteq \mathcal{V}$.
Then $\widehat{\mathcal{K}}$ is a complete and completely distributive lattice, called the *completion* of $\mathcal{K}$. Identifying $a \in \mathcal{K}$ with the upwardly closed set $\mathcal{U}_a = \{x \in \mathcal{K} \mid x \geq a\}$, we see that $\mathcal{K}$ is a subordering of $\widehat{\mathcal{K}}$, namely, $a \leq b$ if and only if $\mathcal{U}_a \leq \mathcal{U}_b$.

For $P \subseteq \mathbb{N}^{\mathbb{N}}$ let $P^* = \{Y \mid (\exists X \in P)\,(X \leq_{\mathsf{T}} Y)\} =$ the Turing upward closure of P. It is easy to check that

$$P \leq_{\mathsf{w}} Q \text{ if and only if } P^* \supseteq Q^*.$$

Thus $\mathcal{D}_{\mathsf{w}} = \widehat{\mathcal{D}_{\mathsf{T}}} =$ the completion of $\mathcal{D}_{\mathsf{T}}$, and Muchnik degrees are identified with upwardly closed sets of Turing degrees.

A digression: suborderings of $\mathcal{D}_{\mathsf{T}}$ and $\mathcal{D}_{\mathsf{w}}$.

Since $\mathcal{D}_{\mathsf{T}}$ and $\mathcal{D}_{\mathsf{w}}$ are large and complicated, it is natural to consider suborderings which are more manageable. Two such suborderings are

$$\mathcal{E}_{\mathsf{T}} = \{\deg_{\mathsf{T}}(\chi_A) \mid A \text{ is a recursively enumerable subset of } \mathbb{N}\}$$

and

$$\mathcal{E}_{\mathsf{w}} = \{\deg_{\mathsf{w}}(P) \mid P \text{ is a nonempty } \Pi^0_1 \text{ subset of } \{0,1\}^{\mathbb{N}}\}.$$

There is a strong analogy between $\mathcal{E}_{\mathsf{T}}$ and $\mathcal{E}_{\mathsf{w}}$.

$\mathcal{E}_{\mathsf{T}}$ is the smallest natural subsemilattice of $\mathcal{D}_{\mathsf{T}}$, and
$\mathcal{E}_{\mathsf{w}}$ is the smallest natural sublattice of $\mathcal{D}_{\mathsf{w}}$.

The bottom and top degrees degrees in $\mathcal{E}_{\mathsf{T}}$ are $\mathbf{0} = \deg_{\mathsf{T}}(0)$ and $\mathbf{0}' = \deg_{\mathsf{T}}(\chi_{0'})$, where $0 = \chi_\emptyset$ and $0' = H =$ the set of Turing machine programs which eventually halt.

The bottom and top degrees in $\mathcal{E}_{\mathsf{w}}$ are $\mathbf{0}$ and $\mathbf{1}$, where $\mathbf{0} = \deg_{\mathsf{w}}(\{0\})$ and $\mathbf{1} = \deg_{\mathsf{w}}(\mathrm{C(PA)})$ where C(PA) is the set of complete and consistent theories which are extensions of first-order Peano arithmetic.

There is a natural semilattice embedding $\mathbf{a} \mapsto \inf(\mathbf{a},\mathbf{1}) : \mathcal{E}_{\mathsf{T}} \hookrightarrow \mathcal{E}_{\mathsf{w}}$ (Simpson 2007). This embedding preserves $\mathbf{0}$ and $\leq$ and sup. However, it does not preserve inf, even when $\inf(\mathbf{a},\mathbf{b})$ exists in $\mathcal{E}_{\mathsf{T}}$.

The most famous structural results for $\mathcal{E}_\mathsf{T}$ are the Splitting Theorem and the Density Theorem.

Splitting Theorem for $\mathcal{E}_\mathsf{T}$ (Sacks 1962). $\mathcal{E}_\mathsf{T}$ satisfies
$\forall x\,(x > 0 \Rightarrow \exists u\,\exists v\,(u < x \text{ and } v < x \text{ and } x = \sup(u,v)))$.

Density Theorem for $\mathcal{E}_\mathsf{T}$ (Sacks 1964). $\mathcal{E}_\mathsf{T}$ satisfies
$\forall x\,\forall y\,(x < y \Rightarrow \exists z\,(x < z < y))$.

There are now analogous results for $\mathcal{E}_\mathsf{w}$:

Splitting Theorem for $\mathcal{E}_\mathsf{w}$ (Binns 2003). $\mathcal{E}_\mathsf{w}$ satisfies
$\forall x\,(x > 0 \Rightarrow \exists u\,\exists v\,(u < x \text{ and } v < x \text{ and } x = \sup(u,v)))$.

Density Theorem for $\mathcal{E}_\mathsf{w}$ (Binns/Shore/Simpson 2014). $\mathcal{E}_\mathsf{w}$ satisfies
$\forall x\,\forall y\,(x < y \Rightarrow \exists z\,(x < z < y))$.

In Hour 3 I will sketch the proof of the Density Theorem for $\mathcal{E}_\mathsf{w}$.

The **Dense Splitting Theorem**
$\forall x\,\forall y\,(x < y \Rightarrow \exists u\,\exists v\,(x < u < y \text{ and } x < v < y \text{ and } y = \sup(u,v)))$
<u>does not hold</u> for $\mathcal{E}_\mathsf{T}$ (Lachlan, Annals of Mathematical Logic, 1976).

It is <u>unknown</u> whether the Dense Splitting Theorem holds for $\mathcal{E}_\mathsf{w}$.

The Muchnik topos.

We may view $\mathcal{D}_{\mathsf{T}}$ as a topological space in which the open sets are the upwardly closed subsets of $\mathcal{D}_{\mathsf{T}}$. Recall also that we have identified the upwardly closed subsets of $\mathcal{D}_{\mathsf{T}}$ with the Muchnik degrees. Therefore, by McKinsey/Tarski 1944, the Muchnik lattice $\mathcal{D}_{\mathsf{w}}$ is a topological model of intuitionistic propositional calculus.

For any topological space $\mathcal{T}$, a *sheaf* over $\mathcal{T}$ consists of a topological space $\mathcal{X}$ together with a local homeomorphism $p : \mathcal{X} \to \mathcal{T}$. A *sheaf morphism* from a sheaf $p : \mathcal{X} \to \mathcal{T}$ to another sheaf $q : \mathcal{Y} \to \mathcal{T}$ is a continuous function $f : \mathcal{X} \to \mathcal{Y}$ such that $p(x) = q(f(x))$ for all $x \in \mathcal{X}$.

Let $\mathrm{Sh}(\mathcal{T})$ = the category of sheaves and sheaf morphisms over $\mathcal{T}$. By Fourman/Scott 1979, $\mathrm{Sh}(\mathcal{T})$ is a topos and a model of intuitionistic higher-order logic. In this model, the truth values are open subsets of $\mathcal{T}$.

Applying the above construction to the topological space $\mathcal{D}_{\mathsf{T}}$, we obtain $\mathrm{Sh}(\mathcal{D}_{\mathsf{T}})$ = the *Muchnik topos*. In this model of intuitionistic mathematics, the truth values are the Muchnik degrees.

We offer $\mathrm{Sh}(\mathcal{D}_{\mathsf{T}})$ as a rigorous implementation of Kolmogorov's 1932 non-rigorous interpretation of intuitionistic mathematics as a "calculus of problems."

The real number system(s) in the Muchnik topos.

Consider the topological space $\mathbb{R}_C = \mathbb{R} \times \mathcal{D}_\mathsf{T}$ with basic open sets $\{x\} \times \mathcal{U}$ where $x \in \mathbb{R}$ and $\mathcal{U} \subseteq \mathcal{D}_\mathsf{T}$ is upwardly closed. There is a projection map $p : \mathbb{R}_C \to \mathcal{D}_\mathsf{T}$ given by $p(x, \mathbf{a}) = \mathbf{a}$. Thus $\mathbb{R}_C$ is a sheaf over $\mathcal{D}_\mathsf{T}$ representing the Cauchy/Dedekind real number system.

An interesting subsheaf of $\mathbb{R}_C$ is $\mathbb{R}_M = \{(x, \mathbf{a}) \in \mathbb{R}_C \mid \deg_\mathsf{T}(x) \leq \mathbf{a}\}$, the sheaf of *Muchnik reals*, which supports an analog of computable analysis. Intuitively, a Cauchy/Dedekind real can exist anywhere within the Turing degrees, but a Muchnik real can exist only where we have enough Turing oracle power to compute it.

Theorem (Basu/Simpson 2014). Let x, y, z be variables ranging over Muchnik reals, let w be a variable ranging over functions from Muchnik reals to Muchnik reals, and let $\Phi(x, y)$ be a formula in which w and z do not occur. Then, the Muchnik topos $\mathrm{Sh}(\mathcal{D}_\mathsf{T})$ satisfies a *Choice and Bounding Principle*

$$(\forall x\, \exists y\, \Phi(x, y)) \Rightarrow (\exists w\, \exists z\, \forall x\, (wx \leq_\mathsf{T} x \oplus z \text{ and } \Phi(x, wx))).$$

Corollary of the proof. If $\mathrm{Sh}(\mathcal{D}_\mathsf{T})$ satisfies $\forall x\, \exists y\, \Phi(x, y)$, then $\mathrm{Sh}(\mathcal{D}_\mathsf{T})$ satisfies $\exists w\, \forall x\, (wx \leq_\mathsf{T} x \text{ and } \Phi(x, wx))$.

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$\mathcal{S}_{\mathsf{w}} =$ the lattice of Muchnik degrees of nonempty Π^0_1 subsets of $\mathbb{N}^{\mathbb{N}}$.

Some specific, natural, Turing degrees.

Given a decision problem $X \in \mathbb{N}^{\mathbb{N}}$, let $X' \in \mathbb{N}^{\mathbb{N}}$ encode the halting problem *relative to* X, i.e., with X used as a Turing oracle.

If $\mathbf{a} = \deg_{\mathsf{T}}(X)$, let $\mathbf{a}' = \deg_{\mathsf{T}}(X')$. It can be shown that $\mathbf{a}'$ is independent of the choice of X such that $\deg_{\mathsf{T}}(X) = \mathbf{a}$. The operator $\mathbf{a} \mapsto \mathbf{a}' : \mathcal{D}_{\mathsf{T}} \to \mathcal{D}_{\mathsf{T}}$ is called the *jump operator*.

Generalizing Turing's proof of unsolvability of the halting problem, we have $\mathbf{a} < \mathbf{a}'$. In other words, the decision problem X' is "more unsolvable than" the decision problem X.

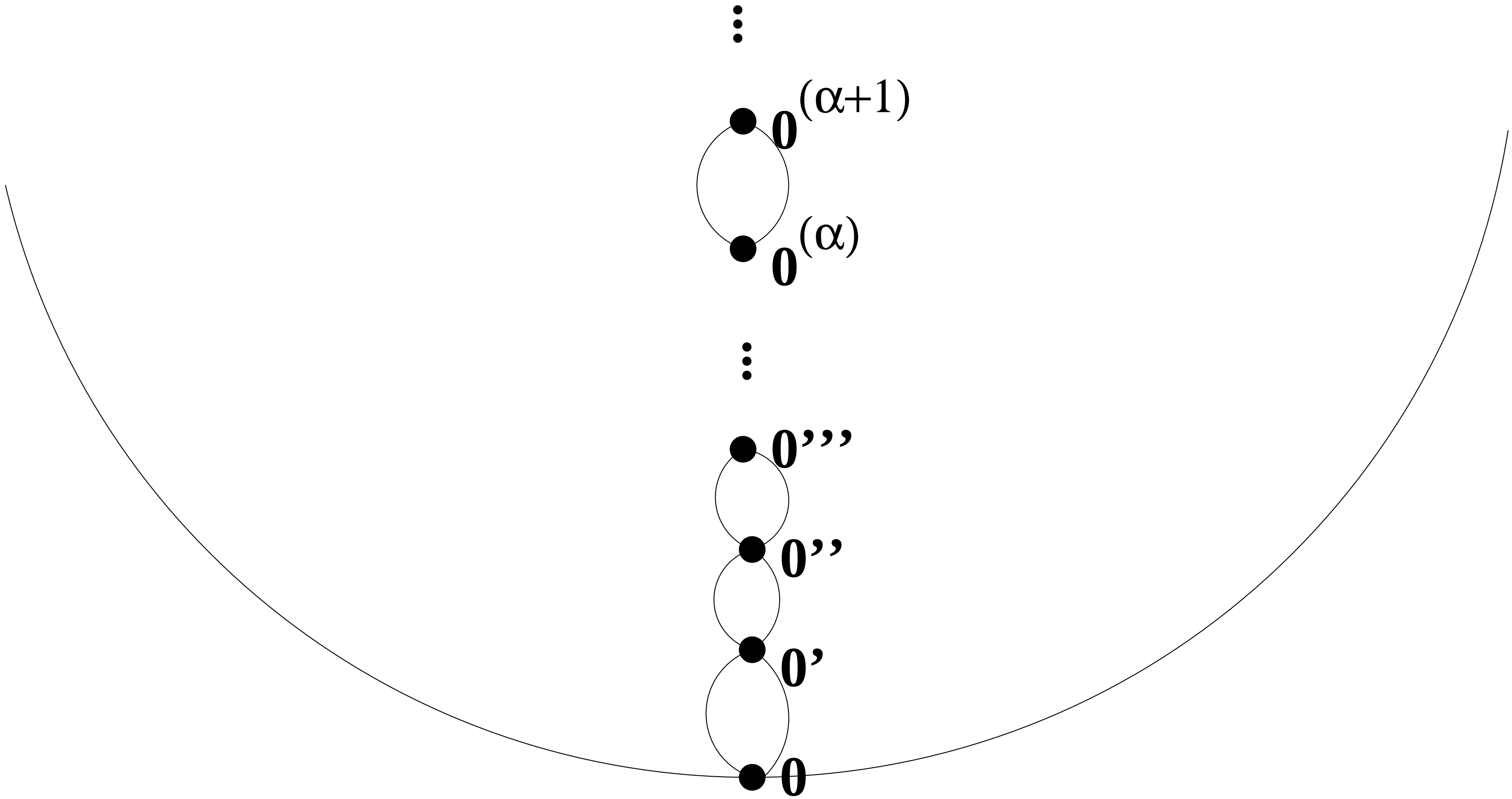
Inductively we define $\mathbf{a}^{(0)} = \mathbf{a}$ and $\mathbf{a}^{(n+1)} = (\mathbf{a}^{(n)})'$ for all $n \in \mathbb{N}$. Extending this induction into the transfinite, we can define $\mathbf{a}^{(\alpha)}$ where α ranges over a large initial segment of the ordinal numbers. The naturalness of this transfinite induction is proved in a series of theorems due to Spector, Sacks, Jockusch/Simpson, and Hodes.

In particular, we have a transfinite sequence of Turing degrees

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \cdots < \mathbf{0}^{(\alpha)} < \mathbf{0}^{(\alpha+1)} < \cdots.$$

Apart from these, no specific natural Turing degrees are known!!!

A picture of $\mathcal{D}_{\mathrm{T}}$, the upper semilattice of Turing degrees.



Apart from the Turing degrees $\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \cdots < \mathbf{0}^{(\alpha)} < \mathbf{0}^{(\alpha+1)} < \cdots$, no <u>specific, natural</u> Turing degrees are known.

A limitation of the Turing degrees.

There are many specific, natural, algorithmically unsolvable problems to which it is impossible to assign a Turing degree.

Example. Let T be a consistent, recursively axiomatizable theory which is effectively essentially undecidable. For instance,
$T = \mathsf{PA} = \mathsf{Z}_1 =$ first-order arithmetic,
or $T = \mathsf{Z}_2 =$ second-order arithmetic,
or $T = \mathsf{ZFC} =$ Zermelo/Fraenkel set theory,
or $T = \mathsf{Q} =$ Robinson's arithmetic,
or $T =$ any consistent, recursively axiomatizable theory which is an extension of one of these.

Any consistent, complete theory which extends T is undecidable.
Let $\mathsf{C}(T)$ be the problem of finding such an extension.
The mass problem $\mathsf{C}(T)$ is specific, natural, and unsolvable, but there is no Turing degree corresponding to $\mathsf{C}(T)$.

The way to overcome this limitation of the Turing degrees is to use mass problems and Muchnik degrees.

Some <u>specific, natural, Muchnik degrees</u>, part 1.

Of course, the specific, natural, Turing degrees
$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \ldots < \mathbf{0}^{(\alpha)} < \mathbf{0}^{(\alpha+1)} < \cdots$
may also be viewed as specific, natural, Muchnik degrees.

Another specific, natural, Muchnik degree is $\mathbf{1} = \deg_{\mathsf{w}}(\mathsf{C}(\mathsf{PA}))$.

Remark. The Muchnik degree $\deg_{\mathsf{w}}(\mathsf{C}(T))$ is independent of our choice of T (so long as T is consistent, recursively axiomatizable, and effectively essentially undecidable). Thus we have
$\mathbf{1} = \deg_{\mathsf{w}}(\mathsf{C}(\mathsf{PA})) = \deg_{\mathsf{w}}(\mathsf{C}(\mathsf{Z}_2)) = \deg_{\mathsf{w}}(\mathsf{C}(\mathsf{ZFC})) = \deg_{\mathsf{w}}(\mathsf{C}(\mathsf{Q}))$.
The Turing degrees $\geq \mathbf{1}$ are often called "PA-degrees," but they could equally well be called "Z_2-degrees" or "ZFC-degrees" or "Q-degrees."

The jump operator applies to Muchnik degrees.
Given $\mathbf{p} = \deg_{\mathsf{w}}(P)$ we define $\mathbf{p}' = \deg_{\mathsf{w}}(\{X' \mid X \in P\})$.

The Kleene Basis Theorem implies that $\mathbf{0} < \mathbf{1} < \mathbf{0}'$.
The Low Basis Theorem implies that $\mathbf{1}' = \mathbf{0}'$.

Some specific, natural, Muchnik degrees, part 2.

Many specific, natural, Muchnik degrees arise from algorithmic randomness and Kolmogorov complexity.

Let $\mathsf{MLR} = \{Z \in \{0,1\}^{\mathbb{N}} \mid Z$ is Martin-Löf random$\}$.

More generally, for $X \in \mathbb{N}^{\mathbb{N}}$ let
$\mathsf{MLR}(X) = \{Z \in \{0,1\}^{\mathbb{N}} \mid Z$ is Martin-Löf random relative to $X\}$.

Let $\mathbf{r}_1 = \deg_{\mathsf{w}}(\mathsf{MLR})$. It is known that $\mathbf{0} < \mathbf{r}_1 < \mathbf{1}$.

Let $\mathbf{r}_\alpha = \deg_{\mathsf{w}}(\bigcap_{\xi<\alpha} \mathsf{MLR}(0^{(\xi)}))$.

Let $\mathbf{b}_\alpha = \deg_{\mathsf{w}}(\{X \in \mathbb{N}^{\mathbb{N}} \mid \mathsf{MLR}(X) \subseteq \mathsf{MLR}(0^{(\alpha)})\})$.

It can be shown that all of these Muchnik degrees are distinct. Clearly the Muchnik degrees $\mathbf{r}_\alpha$ and $\mathbf{b}_\alpha$ are specific and natural, provided the ordinal number α is specific and natural.

Some <u>specific, natural, Muchnik degrees</u>, part 2 (continued).

Remark. The Muchnik degree $\mathbf{r}_1$ is relevant for
the reverse mathematics of measure theory.
The Muchnik degrees $\mathbf{b}_\alpha$ for $\alpha < \omega_1^{\mathsf{CK}}$ are relevant for
the reverse mathematics of measure-theoretic regularity.

Definition. Let $\lambda =$ the fair coin probability measure on $\{0,1\}^{\mathbb{N}}$.
Say that $X \in \mathbb{N}^{\mathbb{N}}$ is *α-regularizing* if for each $\Sigma^0_{\alpha+2}$ set $S_{\alpha+2} \subseteq \{0,1\}^{\mathbb{N}}$
we can find a $\Sigma^{0,X}_2$ set $S^X_2 \subseteq S_{\alpha+2}$ such that $\lambda(S^X_2) = \lambda(S_{\alpha+2})$.

Theorem (Simpson 2008). $\mathbf{b}_\alpha = \deg_{\mathsf{w}}(\{X \mid X \text{ is } \alpha\text{-regularizing}\})$.

For $\alpha = 1$ this is due to Kjos-Hanssen/Miller/Solomon 2006.

Some specific, natural, Muchnik degrees, part 3.

Given $f:\mathbb{N}\to\mathbb{N}$, say that $Z\in\{0,1\}^{\mathbb{N}}$ is *strongly f-complex* if $\exists c\,\forall n\,(\mathsf{KA}(Z{\upharpoonright}n)\geq f(n)-c)$. In other words, f specifies a lower bound for the a priori Kolmogorov complexity of the first n bits of Z.

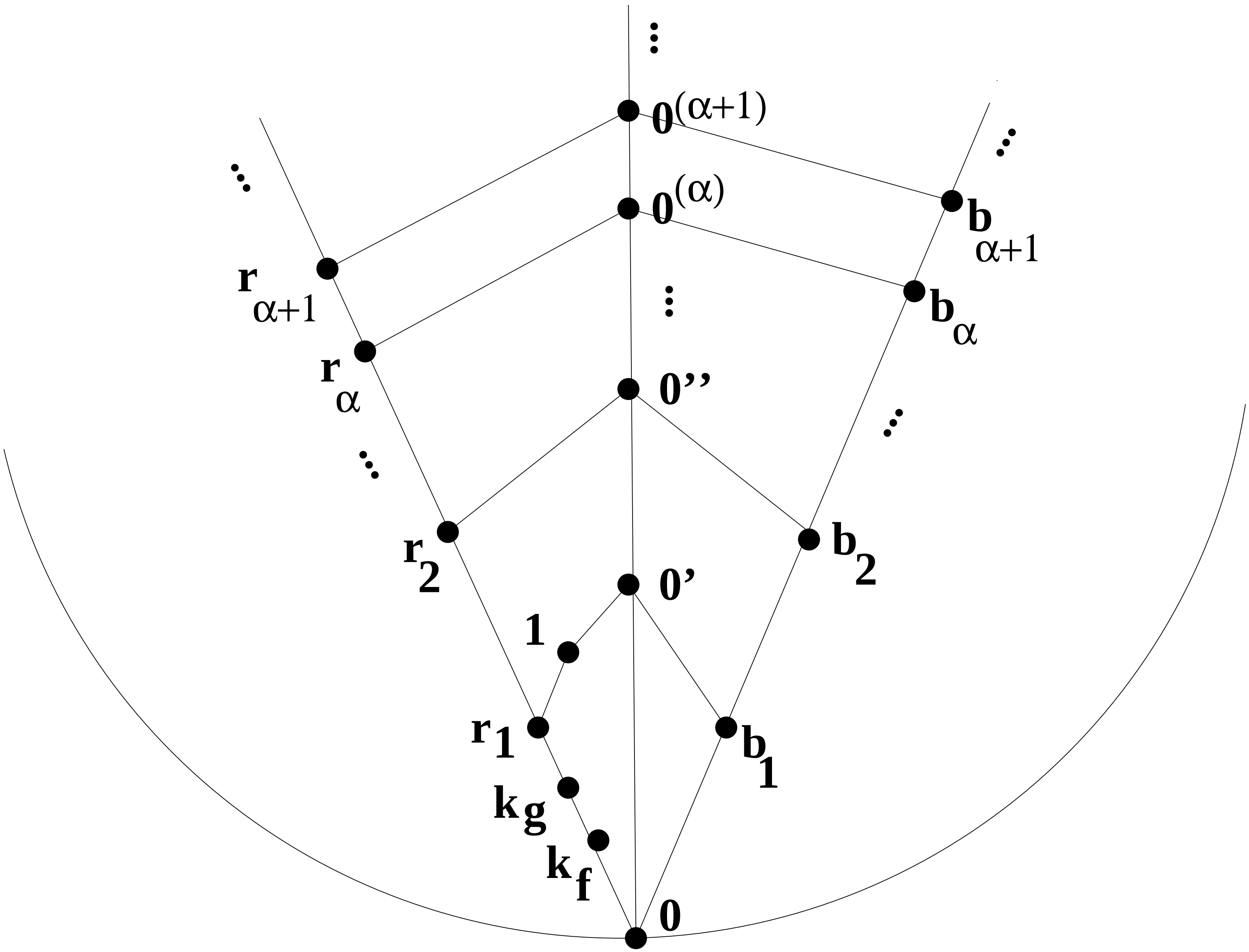
Let $\mathbf{k}_f=\deg_{\mathsf{w}}(\{Z\in\{0,1\}^{\mathbb{N}}\mid Z\text{ is strongly }f\text{-complex}\})$.

Clearly the Muchnik degree $\mathbf{k}_f$ is specific and natural, provided f is specific and natural. Also, by Schnorr's Theorem, we have $\mathbf{k}_1=\mathbf{r}_1$ where the 1 in $\mathbf{k}_1$ denotes the identity function.

It is known that $\mathbf{k}_f<\mathbf{k}_g\leq\mathbf{r}_1$ holds for many pairs $f,g:\mathbb{N}\to\mathbb{N}$. In particular, it holds when f and g are recursive functions such that $\forall n\,(f(n)\leq f(n+1)\leq f(n)+1$ and $f(n)+2\log_2 f(n)\leq g(n)\leq n)$. This result is due to Hudelson 2014 building on Miller 2011.

Examples. Let $f(n)=n/3$ and $g(n)=n/2$,
or let $f(n)=\sqrt[3]{n}$ and $g(n)=\sqrt[2]{n}$,
or let $f(n)=\log_3 n$ and $g(n)=\log_2 n$,
or let $f(n)=\log_2 n$ and $g(n)=\log_2 n+2\log_2\log_2 n$,
or let $f(n)=n-2\log_2 n$ and $g(n)=n$.

A picture of $\mathcal{D}_{\mathrm{w}}$, the lattice of Muchnik degrees.



Some specific, natural, Muchnik degrees, part 4.

There are many more examples of specific, natural Muchnik degrees.

Definition. A partial recursive function $\psi :\subseteq \mathbb{N} \to \mathbb{N}$ is said to be *linearly universal* if for each partial recursive function $\varphi :\subseteq \mathbb{N} \to \mathbb{N}$ there exist $a, b \in \mathbb{N}$ such that $\forall n\, (\varphi(n) \simeq \psi(an + b))$.

An example of such a function is $\psi(2^e(2n + 1)) \simeq \varphi_e(n)$.

Let $\mathbf{d} = \deg_{\mathsf{w}}(D)$ where $D = \{Z \in \mathbb{N}^{\mathbb{N}} \mid \forall n\, (Z(n) \not\simeq \psi(n))$ for some linearly universal, partial recursive function $\psi\}$.

Let $\mathbf{d}_{\mathsf{REC}} = \deg_{\mathsf{w}}(\{Z \in D \mid Z$ is recursively bounded$\})$.

It is known that $\mathbf{0} < \mathbf{d}_{\mathsf{REC}} < \mathbf{d} < \mathbf{r}_1$ (Ambos-Spies et al, 2004).

Remark. Clearly $\mathbf{d} = \deg_{\mathsf{w}}(\{Z \in \mathbb{N}^{\mathbb{N}} \mid Z$ is diagonally nonrecursive$\})$, and $\mathbf{d}_{\mathsf{REC}} = \deg_{\mathsf{w}}(\{Z \mid Z$ diagonally nonrecursive, recursively bounded$\})$. However, the definition of $\mathbf{d}$ and $\mathbf{d}_{\mathsf{REC}}$ in terms of linear universality is preferable when it comes to refinements in terms of growth rates.

See the theorem on the next slide.

Some specific, natural, Muchnik degrees, part 4 (continued).

Recall that $D = \{Z \in \mathbb{N}^{\mathbb{N}} \mid \forall n\,(Z(n) \not\simeq \psi(n))$
for some linearly universal, partial recursive function $\psi\}$.

Definition. For $h : \mathbb{N} \to \mathbb{N}$ let $\mathbf{d}_h = \deg_{\mathsf{w}}(\{Z \in D \mid \forall n\,(Z(n) < h(n))\})$.

Remark. If h is bounded and $\forall n\,(2 \le h(n))$, then $\mathbf{d}_h = \mathbf{1}$.

Theorem (Greenberg/Miller 2011; Miller)**.** Let h be
an unbounded recursive function such that $\forall n\,(2 \le h(n) \le h(n+1))$.

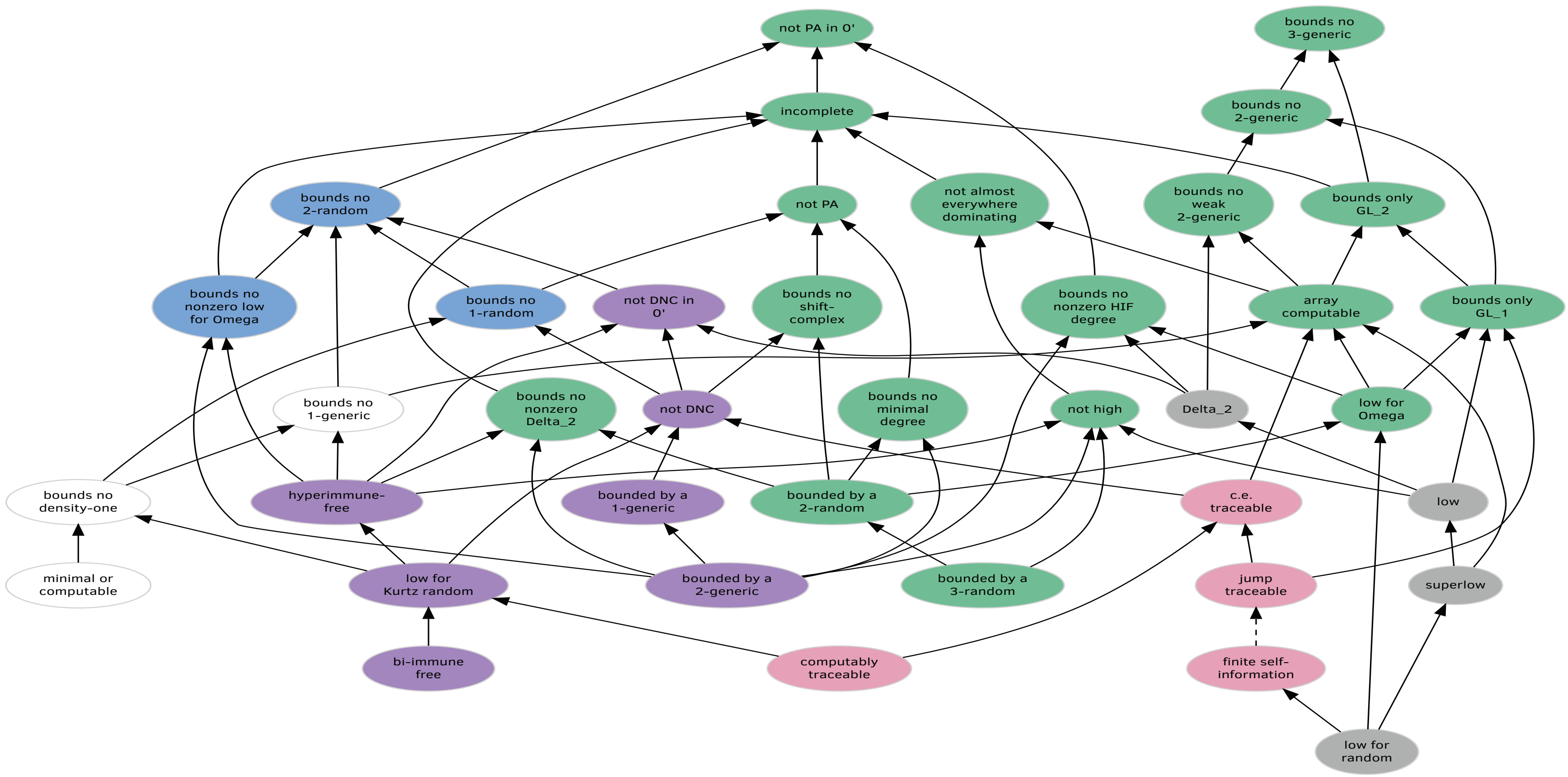
1. $\mathbf{d}_{\mathsf{REC}} < \mathbf{d}_h < \mathbf{1}$.
2. If $\sum_n h(n)^{-1} < \infty$ then $\mathbf{d}_h < \mathbf{r}_1$.
3. If $\sum_n h(n)^{-1} = \infty$ then $\mathbf{d}_h$ is incomparable with $\mathbf{r}_\alpha$ for all $\alpha \ge 1$.

Remark. The degrees $\mathbf{d}_h$ where h is as in 2 above are closely intertwined with the degrees $\mathbf{k}_f$ where f is an unbounded recursive function such that $\forall n\,(f(n) \le n)$. In particular we have $\mathbf{d}_{\mathsf{REC}} = \mathbf{k}_{\mathsf{REC}}$ where $\mathbf{k}_{\mathsf{REC}}$ is the infimum of the $\mathbf{k}_f$'s for all such f.

It would be nice to have a more precise hierarchy theorem for the $\mathbf{d}_h$'s which would be analogous to Hudelson's theorem for the $\mathbf{k}_f$'s.

Another picture of $\mathcal{D}_{\mathrm{w}}$, the lattice of Muchnik degrees.

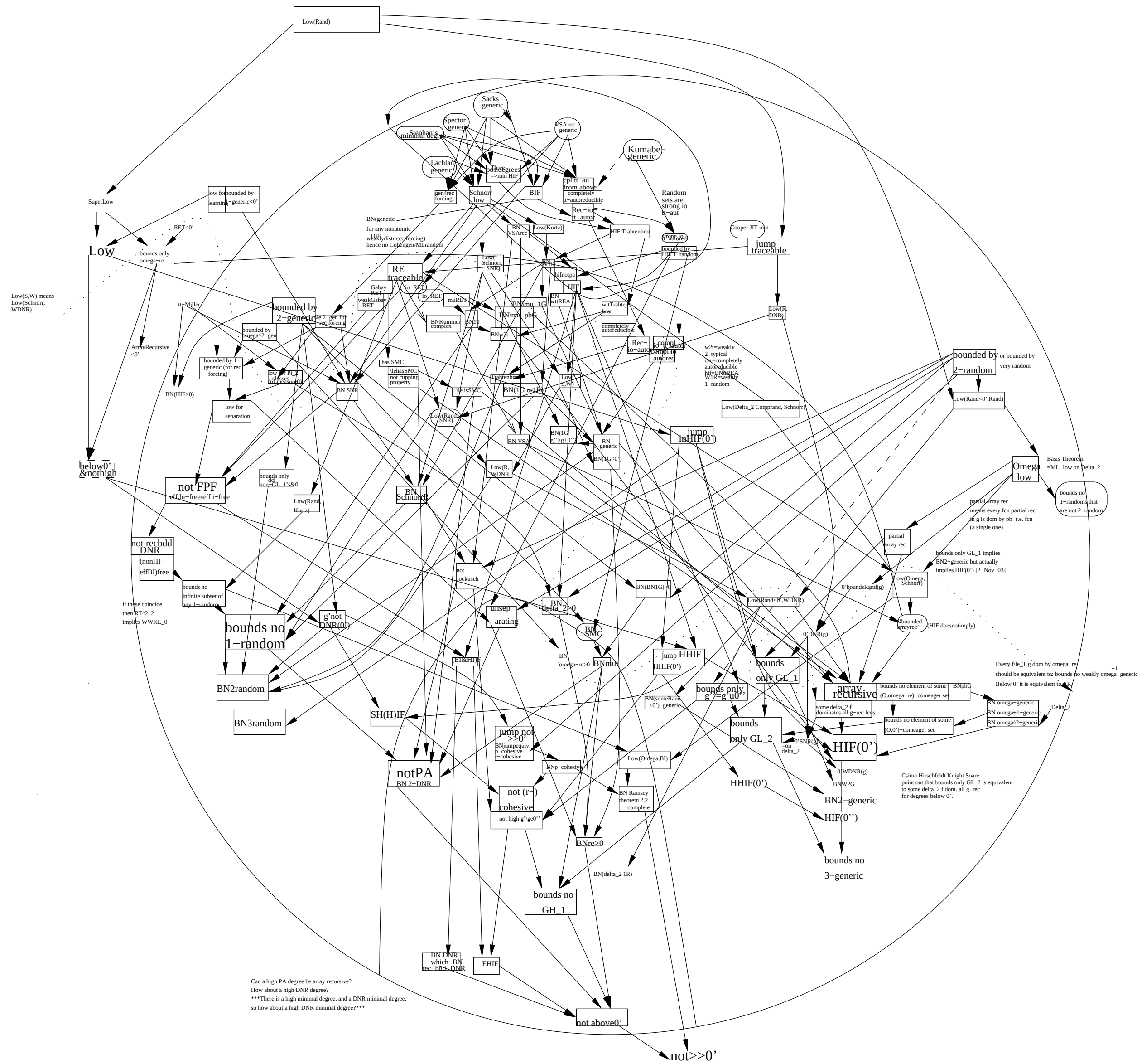
Each oval represents a specific, natural, Muchnik degree.



Originally this picture was intended to represent the Computability Menagerie, as developed by Bjørn Kjos-Hanssen, Joseph S. Miller, and Mushfeq Khan. The inhabitants of the menagerie are downwardly closed sets of Turing degrees. The complements of these sets are upwardly closed sets of Turing degrees, i.e., Muchnik degrees. So this is also a picture of the Muchnik degrees. The picture itself is courtesy of Joseph S. Miller.

Yet another picture of $\mathcal{D}_\mathrm{w}$, the lattice of Muchnik degrees.

Each box represents a specific, natural, Muchnik degree.



This picture is courtesy of Bjørn Kjos-Hanssen.

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$\mathcal{E}_{\mathrm{T}}$ = **the upper semilattice of r. e. Turing degrees.**

$\mathcal{E}_{\mathrm{w}}$ = **the lattice of Muchnik degrees of Π^0_1 sets $\neq \emptyset$ in $\{0,1\}^{\mathbb{N}}$.**

$\mathcal{S}_{\mathrm{w}}$ = **the lattice of Muchnik degrees of Π^0_1 sets $\neq \emptyset$ in $\mathbb{N}^{\mathbb{N}}$.**

The sublattices $\mathcal{E}_\mathsf{w}$ and $\mathcal{S}_\mathsf{w}$.

Since $\mathcal{D}_\mathsf{w}$ is large and complicated, it is natural to consider sublattices which are more manageable. Two such sublattices are

$$\mathcal{E}_\mathsf{w} = \{\deg_\mathsf{w}(P) \mid \emptyset \neq P \subseteq \{0,1\}^\mathbb{N} \text{ and } P \text{ is } \Pi^0_1\}$$

and

$$\mathcal{S}_\mathsf{w} = \{\deg_\mathsf{w}(P) \mid \emptyset \neq P \subseteq \mathbb{N}^\mathbb{N} \text{ and } P \text{ is } \Pi^0_1\}\,.$$

We compare $\mathcal{E}_\mathsf{w}$ to $\mathcal{E}_\mathsf{T}$ = the upper semilattice of r.e. Turing degrees.

There is a strong analogy between $\mathcal{E}_\mathsf{w}$ and $\mathcal{E}_\mathsf{T}$:

(a) $\mathcal{E}_\mathsf{w}$ is the smallest natural sublattice of $\mathcal{D}_\mathsf{w}$, just as $\mathcal{E}_\mathsf{T}$ is the smallest natural subsemilattice of $\mathcal{D}_\mathsf{T}$.

(b) There is a natural embedding $\mathbf{a} \mapsto \inf(\mathbf{a}, \mathbf{1}) : \mathcal{E}_\mathsf{T} \hookrightarrow \mathcal{E}_\mathsf{w}$.

(c) The Splitting Theorem and the Density Theorem, due to Sacks for $\mathcal{E}_\mathsf{T}$, also hold for $\mathcal{E}_\mathsf{w}$. See below.

However, $\mathcal{E}_\mathsf{w}$ has an advantage over $\mathcal{E}_\mathsf{T}$:

$\mathcal{E}_\mathsf{w}$ contains many specific, natural degrees associated with specific, natural, foundationally interesting problems. In contrast, $\mathcal{E}_\mathsf{T}$ is not known to contain any such degrees other than $\mathbf{0}'$ and $\mathbf{0}$.

Some facts about $\mathcal{E}_\mathsf{w}$ and $\mathcal{S}_\mathsf{w}$.

Fact 1. The bottom and top degrees in $\mathcal{E}_\mathsf{w}$ are $\mathbf{0}$ and $\mathbf{1}$ respectively. The bottom degree in $\mathcal{S}_\mathsf{w}$ is $\mathbf{0}$, but there is no top degree in $\mathcal{S}_\mathsf{w}$.

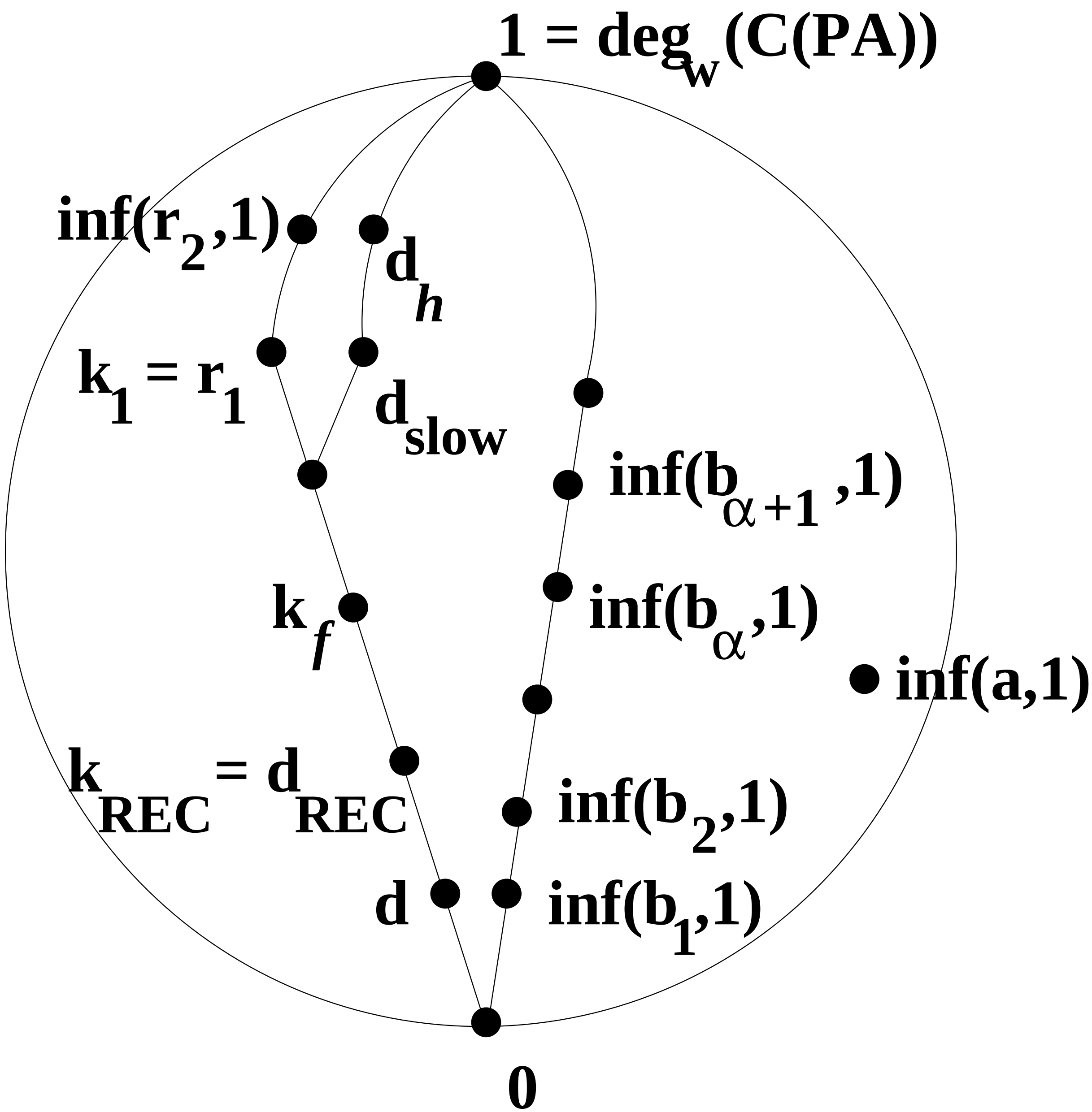
Fact 2. $\mathcal{S}_\mathsf{w} = \{\deg_\mathsf{w}(S) \mid \emptyset \neq S \subseteq \mathbb{N}^\mathbb{N}$ and S is $\mathbf{\Sigma}^0_3\}$.

This is important because it implies that many specific, natural, Muchnik degrees belong to $\mathcal{S}_\mathsf{w}$. Examples:

- $\mathbf{0}^{(\alpha)}, \mathbf{b}_\alpha \in \mathcal{S}_\mathsf{w}$ for all recursive ordinal numbers α.
- $\mathbf{r}_1, \mathbf{r}_2, \mathbf{k}_f \in \mathcal{S}_\mathsf{w}$ for all recursive $f : \mathbb{N} \to \mathbb{N}$ satisfying $\forall n\,(f(n) \leq n)$.
- $\mathbf{d}, \mathbf{d}_\mathsf{REC}, \mathbf{d}_h \in \mathcal{S}_\mathsf{w}$ for all recursive $h : \mathbb{N} \to \mathbb{N}$ satisfying $\forall n\,(2 \leq h(n))$.

Fact 3 (Simpson 2007). $\mathcal{E}_\mathsf{w}$ is an initial segment of $\mathcal{S}_\mathsf{w}$.

This is important because it gives us a specific, natural, lattice homomorphism $\mathbf{s} \mapsto \inf(\mathbf{s}, \mathbf{1}) : \mathcal{S}_\mathsf{w} \to \mathcal{E}_\mathsf{w}$. This homomorphism carries all of the specific, natural, Muchnik degrees in $\mathcal{S}_\mathsf{w}$ to specific, natural, Muchnik degrees in $\mathcal{E}_\mathsf{w}$. Hence $\mathcal{E}_\mathsf{w}$ contains many such degrees.



This is a picture of $\mathcal{E}_\mathrm{w}$. Each black dot except inf$(\mathbf{a}, 1)$ represents a specific, natural, Muchnik degree in $\mathcal{E}_\mathrm{w}$.

Proof of Fact 3.

Fact 3 says that $\mathcal{E}_\mathsf{w}$ is an initial segment of $\mathcal{S}_\mathsf{w}$.

To prove Fact 3, it suffices to prove:

> Given nonempty Π^0_1 sets $P \subseteq \{0,1\}^\mathbb{N}$ and $S \subseteq \mathbb{N}^\mathbb{N}$,
> we can find a nonempty Π^0_1 set $Q \subseteq \{0,1,2\}^\mathbb{N}$
> such that $\deg_\mathsf{w}(Q) = \inf(\deg_\mathsf{w}(P), \deg_\mathsf{w}(S))$.

To prove this, let $U \subseteq \{0,1\}^*$ and $V \subseteq \mathbb{N}^*$ be computable trees such that $P = \{\text{paths through } U\}$ and $S = \{\text{paths through } V\}$.

Let $Q = \{\text{paths through } W\}$ where $W \subseteq \{0,1,2\}^*$ is the computable tree consisting of all sequences of the form

$$\sigma_1{}^\frown\langle 2\rangle^\frown \cdots {}^\frown\langle 2\rangle^\frown\sigma_{n-1}{}^\frown\langle 2\rangle^\frown\sigma_n$$

with $n \geq 1$ and $\sigma_1, \ldots, \sigma_{n-1}, \sigma_n \in U$ and $\langle |\sigma_1|, \ldots, |\sigma_{n-1}|\rangle \in V$.

It is easy to check that this works.

The Splitting and Density Theorems for $\mathcal{E}_{\mathsf{w}}$.

Splitting Theorem (Binns 2003). $\mathcal{E}_{\mathsf{w}}$ satisfies the Splitting Theorem: $\forall x\,(x > 0 \Rightarrow \exists u\,\exists v\,(u < x \text{ and } v < x \text{ and } x = \sup(u, v)))$.

Density Theorem (Binns/Shore/Simpson 2014). $\mathcal{E}_{\mathsf{w}}$ satisfies the Density Theorem: $\forall x\,\forall y\,(x < y \Rightarrow \exists z\,(x < z < y))$.

We now sketch the proof that $\mathcal{E}_{\mathsf{w}}$ is dense. Since $\mathcal{E}_{\mathsf{w}}$ is an initial segment of $\mathcal{S}_{\mathsf{w}}$, it will suffice to prove that $\mathcal{S}_{\mathsf{w}}$ is dense.

The proof will be presented in a modular way, with several lemmas.

Lemma 1. Let $Q \subseteq \mathbb{N}^{\mathbb{N}}$ be Π^0_1 such that $Q \not\leq_{\mathsf{w}} \{0\}$. Then for all $Y \in \mathbb{N}^{\mathbb{N}}$ there exists $\widehat{Y} \in \mathbb{N}^{\mathbb{N}}$ such that $0' \oplus Y \equiv_{\mathsf{T}} 0' \oplus \widehat{Y} \equiv_{\mathsf{T}} \widehat{Y}'$ and $Q \not\leq_{\mathsf{w}} \{\widehat{Y}\}$.

Lemma 1 is proved like the Friedberg Jump Theorem, with extra steps taken to insure that $Q \not\leq_{\mathsf{w}} \{\widehat{Y}\}$.

Lemma 2. Given Π^0_1 predicates $U, V \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$, we can find a Π^0_1 predicate $\widehat{U} \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ such that for each X with $\{Z \mid V(X,Z)\} \not\leq_{\mathrm{w}} \{X\}$ there is a homeomorphism $Y \mapsto \widehat{Y}$ of $\{Y \mid U(X,Y)\}$ onto $\{\widehat{Y} \mid \widehat{U}(X,\widehat{Y})\}$ such that $X' \oplus Y \equiv_{\mathrm{T}} X' \oplus \widehat{Y} \equiv_{\mathrm{T}} (X \oplus \widehat{Y})'$ and $\{Z \mid V(X,Z)\} \not\leq_{\mathrm{w}} \{X \oplus \widehat{Y}\}$.

Lemma 2 is proved by uniformly relativizing Lemma 1 to X, taking extra care to insure that $\{\widehat{Y} \mid \widehat{U}(X,\widehat{Y})\}$ is uniformly Π^0_1 relative to X.

Lemma 3. Suppose Kleene's O is not hyperarithmetical in X. Then, there is a nonempty Π^0_1 set $S \subseteq \mathbb{N}^\mathbb{N}$ such that $S \not\leq_{\mathrm{w}} \{X'\}$.

Lemma 3 follows from the Kleene Normal Form Theorem plus the fact that Kleene's O is Π^1_1.

We now prove that $\mathcal{S}_{\mathrm{w}}$ is dense. Given Π^0_1 sets $P, Q \subseteq \mathbb{N}^\mathbb{N}$ such that $P <_{\mathrm{w}} Q$, to find a Π^0_1 set $R \subseteq \mathbb{N}^\mathbb{N}$ such that $P <_{\mathrm{w}} R <_{\mathrm{w}} Q$. By the Gandy Basis Theorem, let $X_0 \in P$ be such that Kleene's O is not hyp. in X_0. By Lemma 3 let $S \subseteq \mathbb{N}^\mathbb{N}$ be nonempty Π^0_1 such that $S \not\leq_{\mathrm{w}} \{X_0'\}$. Apply Lemma 2 with $U(X,Y) \equiv Y \in S$ and $V(X,Z) \equiv Z \in Q$. Let $R = \{X \oplus \widehat{Y} \mid X \in P \text{ and } \widehat{U}(X,\widehat{Y})\} \cup Q$ where $\widehat{U}$ is as in the conclusion of Lemma 2. It is easy to check that this works.

Details of the construction for Lemmas 1 and 2.

We give the unrelativized construction, with $X = 0$.
In presenting the construction, we do not assume $Q \not\leq_{\mathrm{w}} \{0\}$.

Let V be a recursive tree such that $Q = \{\text{paths through } V\}$.

To each string σ we associate an infinite sequence of strings $\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_s \subseteq \tau_{s+1} \subseteq \cdots$. Later we shall write $F_s(\sigma) = \tau_s$.

Stage 0. Let $\tau_0 = \langle\rangle$ and $i_0 = 1$ and $n_0 = 0$.

Stage $s+1$. Let $n = n_s$. If $n \geq |\sigma|$ let $\tau_t = \tau_s$ and $i_t = i_s$ and $n_t = n_s$ for all $t \geq s+1$. If $n < |\sigma|$ we proceed depending on the value of i_s.

Case 1: $i_s = 1$. Let $\tau_{s+1} = \tau_s{}^\frown\langle\sigma(n)\rangle$ and $i_{s+1} = 2$ and $n_{s+1} = n_s$.

Case 2: $i_s = 2$. If $(\exists\tau \supseteq \tau_s)\,(\{n\}^{\tau}_{|\tau|}(n)\downarrow)$ let $\tau_{s+1} =$ the least such τ, otherwise let $\tau_{s+1} = \tau_s$. Either way let $i_{s+1} = 3$ and $n_{s+1} = n_s$.

Case 3: $i_s = 3$. If $(\exists\tau \supset \tau_s)\,(\{n\}^{\tau_s} \subset \{n\}^{\tau} \in V)$ let $\tau_{s+1} =$ the least such τ, and let $i_{s+1} = 3$ and $n_{s+1} = n_s$.
Otherwise let $\tau_{s+1} = \tau_s$ and $i_{s+1} = 1$ and $n_{s+1} = n_s + 1$.

This completes the definition of $\tau_s = F_s(\sigma)$.

Note that $F_s : \sigma \mapsto \tau_s : \mathbb{N}^* \to \mathbb{N}^*$ is uniformly $\leq_{\mathrm{T}} 0'$ and monotone, i.e., $\rho \subseteq \sigma$ and $s \leq t$ imply $F_s(\rho) \subseteq F_t(\sigma)$.

Define $Y \mapsto \widetilde{Y} : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by $\widetilde{Y} = F(Y) = \bigcup_m \bigcup_s F_s(Y{\restriction}m)$.

Given a Π^0_1 set $U \subseteq \mathbb{N}^{\mathbb{N}}$, consider the $\Pi^{0,0'}_3$ set $\widetilde{U} = \{\widetilde{Y} \mid Y \in U\}$.

For each $Y \in U$ and each $n \in \mathbb{N}$ we have $\widetilde{Y}{\restriction}n \subseteq F_s(Y{\restriction}m)$ for some $s \leq 3n$ and some m such that $Y{\restriction}m$ is a *substring* of $\widetilde{Y}{\restriction}n$, i.e., $Y{\restriction}m = \langle \widetilde{Y}(j_1), \ldots, \widetilde{Y}(j_m) \rangle$ for some $j_1 < \cdots < j_m < n$. Therefore, in the $\Pi^{0,0'}_3$ definition of $\widetilde{U}$, the unbounded existential quantifiers may be replaced by bounded ones. Thus $\widetilde{U}$ is actually $\Pi^{0,0'}_1$, hence Π^0_2, say $\widetilde{U} = \{\widetilde{Y} \mid \forall i \, \exists j \, R(\widetilde{Y}, i, j)\}$ where R is recursive. Our Π^0_1 set is then $\widehat{U} = \{\widetilde{Y} \oplus \widetilde{Y}^* \mid \forall i \, (\widetilde{Y}^*(i) =$ the least j such that $R(\widetilde{Y}, i, j)$ holds$)\}$.

Assume now that $Q \not\leq_{\mathrm{w}} \{0\}$. In this situation, our construction is just the standard proof of the Friedberg Jump Theorem, with extra steps (Case 3) to insure that $Q \not\leq_{\mathrm{w}} \{\widetilde{Y}\}$. Thus $0' \oplus Y \equiv_{\mathrm{T}} 0' \oplus \widetilde{Y}' \equiv_{\mathrm{T}} \widetilde{Y}'$ and $Y \mapsto \widetilde{Y}$ is a homeomorphism of U onto $\widetilde{U}$. For each $Y \in U$ let $\widetilde{Y}^*(i) =$ the least j such that $R(\widetilde{Y}, i, j)$ holds. Then $\widetilde{Y}^* \leq_{\mathrm{T}} \widetilde{Y}$, so $\widehat{Y} = \widetilde{Y} \oplus \widetilde{Y}^*$ has the same properties as $\widetilde{Y}$, i.e., $Q \not\leq_{\mathrm{w}} \{\widehat{Y}\}$ and $0' \oplus Y \equiv_{\mathrm{T}} 0' \oplus \widehat{Y} \equiv_{\mathrm{T}} \widehat{Y}'$ and $Y \mapsto \widehat{Y}$ is a homeomorphism of U onto $\widehat{U}$.

This completes the proof!

Summary of this 3-hour tutorial.

1. $\mathcal{D}_{\mathsf{T}}$ = the semilattice of Turing degrees.
2. $\mathcal{E}_{\mathsf{T}}$ = the semilattice of recursively enumerable Turing degrees.
3. $\mathcal{D}_{\mathsf{w}} = \widehat{\mathcal{D}_{\mathsf{T}}}$ = the lattice of Muchnik degrees.
4. $\mathcal{E}_{\mathsf{w}}$ = the lattice of Muchnik degrees of nonempty Π^0_1 sets in $\{0,1\}^{\mathbb{N}}$.
5. There is a natural embedding of $\mathcal{D}_{\mathsf{T}}$ into its completion $\mathcal{D}_{\mathsf{w}}$.
6. There is a natural embedding of $\mathcal{E}_{\mathsf{T}}$ into $\mathcal{E}_{\mathsf{w}}$.
7. The Splitting and Density Theorems hold for $\mathcal{E}_{\mathsf{T}}$ and for $\mathcal{E}_{\mathsf{w}}$.
8. There is a strong analogy between $\mathcal{E}_{\mathsf{T}}$ and $\mathcal{E}_{\mathsf{w}}$.
9. In $\mathcal{D}_{\mathsf{T}}$ the only known specific, natural, degrees are among $\mathbf{0}, \mathbf{0}', \mathbf{0}'', \ldots, \mathbf{0}^{(\alpha)}, \mathbf{0}^{(\alpha+1)}, \ldots$.
10. In $\mathcal{D}_{\mathsf{w}}$ there are many other specific, natural degrees including $\mathbf{r}_\alpha$'s and $\mathbf{b}_\alpha$'s.
11. In $\mathcal{E}_{\mathsf{T}}$ the only known specific, natural degrees are $\mathbf{0}$ and $\mathbf{0}'$.
12. In $\mathcal{E}_{\mathsf{w}}$ there are many specific, natural degrees including $\mathbf{1}, \mathbf{r}_1 = \mathbf{k}_1$, $\mathbf{k} = \mathbf{d}, \mathbf{k}_{\mathsf{REC}} = \mathbf{d}_{\mathsf{REC}}, \mathbf{k}_f, \mathbf{d}_h, \mathbf{d}_{\mathsf{slow}}, \inf(\mathbf{r}_2, \mathbf{1}), \inf(\mathbf{b}_\alpha, \mathbf{1})$ where $\alpha < \omega_1^{\mathsf{CK}}$.

Thank you for your attention!