Measure-Theoretic Regularity, Degrees of Unsolvability, and Reverse Mathematics

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NSF-DMS-0600823, NSF-DMS-0652637,
Grove Endowment, Templeton Foundation

Reverse Mathematics Workshop

University of Chicago

November 6–8, 2009

A set in Euclidean space is F_σ if it is the union of a countable sequence of closed sets. Also, the Borel sets are obtained from closed sets by repeatedly taking complements and countable unions.

Measure-theoretic regularity: Every Lebesgue measurable set includes an F_σ set of the same measure. In particular, every Borel set includes an F_σ set of the same measure.

We investigate metamathematical aspects of measure-theoretic regularity.

Degrees of unsolvability: We quantify the "descriptive complexity" or "computational strength" of the F_σ sets which are needed in order to implement measure-theoretic regularity at various levels of the effective Borel hierarchy.

Reverse mathematics: We calibrate the "logical strength" of various statements of measure-theoretic regularity.

Instead of Euclidean space, it is convenient to use the Cantor space, $\{0,1\}^{\mathbb{N}}$.

The effective Borel hierarchy:

Let S be a subset of $\{0,1\}^{\mathbb{N}}$. S is Σ^0_0 if and only if S is clopen. These sets are indexed in an obvious way. For each recursive ordinal α, S is $\Sigma^0_{\alpha+1}$ if and only if $S = \bigcup_{i=0}^{\infty} P_i$ where P_i is Π^0_α with an index which is computable as a function of i. An index of S is an index of this computable function. S is Π^0_α if its complement is Σ^0_α. For limit ordinals α, S is Σ^0_α if and only if S is Σ^0_β for some $\beta < \alpha$.

This definition relativizes to an arbitrary Turing oracle X.

Remark. S is Borel if and only if S is $\Sigma^{0,X}_\alpha$ for some Turing oracle X and some $\alpha < \omega_1^X$. S is F_σ if and only if S is $\Sigma^{0,X}_2$ for some X.

Let $X^{(\alpha)}$ denote the αth Turing jump of X.

The *fair coin measure* on $\{0,1\}^{\mathbb{N}}$ is given by $\mu(N_\sigma) = 1/2^{|\sigma|}$. Here σ is a *bitstring*, i.e., a finite sequence of 0's and 1's.
We are writing $N_\sigma = \{Z \in \{0,1\}^{\mathbb{N}} \mid \sigma \subset Z\}$ and $|\sigma| =$ the length of σ.

Lemma. Every $\Sigma^{0,X}_{\alpha+2}$ set includes a $\Sigma^{0,X^{(\alpha)}}_{2}$ set of the same measure. Conversely, every $\Sigma^{0,X^{(\alpha)}}_{2}$ set is $\Sigma^{0,X}_{\alpha+2}$.

Proof. For $\alpha = n$ see Steven M. Kautz, PhD thesis, Cornell, 1991. The generalization to arbitrary $\alpha < \omega_1^X$ is routine.

Definition (Nies)**.**

1. $X \leq_{\mathsf{LR}} Y$ if $\mathsf{R}^Y \subseteq \mathsf{R}^X$. Here $\mathsf{R}^X = \{Z \mid Z$ is Martin-Löf random rel. to $X\}$.

2. $X \leq_{\mathsf{LK}} Y$ if $\mathsf{K}^Y(\tau) \leq \mathsf{K}^X(\tau) + O(1)$ for all bitstrings τ. Here $\mathsf{K}^X(\tau) =$ the prefix-free Kolmogorov complexity of τ relative to X.

Remark. $X \leq_{\mathsf{T}} Y$ implies $X \leq_{\mathsf{LR}} Y$, but the converse fails badly.

Theorem. The following are equivalent.

1. $X \leq_{\mathsf{LR}} Y$.
2. $X \leq_{\mathsf{LK}} Y$.
3. Every $\Sigma_2^{0,X}$ set of positive measure includes a $\Sigma_2^{0,Y}$ set of positive measure.

Theorem. The following are equivalent.

1. $X \leq_{\mathsf{LR}} Y$ and $X \leq_{\mathsf{T}} Y'$.
2. Every $\Sigma_2^{0,X}$ set includes a $\Sigma_2^{0,Y}$ set of the same measure.

These results are due to Kjos-Hanssen and Kjos-Hanssen/Miller/Solomon in 2005 and 2006 building on earlier work of Dobrinen/Simpson concerning almost everywhere domination and measure-theoretic regularity.

Thus LR-reducibility is closely related to measure-theoretic regularity.

Summarizing the above results, we have:

Theorem. The following are equivalent.

1. $X^{(\alpha)} \leq_{\mathsf{LR}} Y$ and $X^{(\alpha)} \leq_{\mathsf{T}} Y'$.
2. Every $\Sigma^{0,X}_{\alpha+2}$ set includes a $\Sigma^{0,Y}_{2}$ set of the same measure.

Recently we improved this to:

Theorem (Simpson 2009)**.**
The following are equivalent.

1. $X^{(\alpha)} \leq_{\mathsf{LR}} Y$ and $X \leq_{\mathsf{T}} Y'$.
2. Every $\Sigma^{0,X}_{\alpha+2}$ set includes a $\Sigma^{0,Y}_{2}$ set of the same measure.

In particular:

Theorem (Simpson 2009)**.**
The following are equivalent.

1. $0^{(\alpha)} \leq_{\mathsf{LR}} Y$.
2. Every $\Sigma^{0}_{\alpha+2}$ set includes a $\Sigma^{0,Y}_{2}$ set of the same measure.

These recent improvements are based on the following technical lemmas concerning LR-reducibility.

Lemma (Simpson 2009)**.**
Suppose $A \leq_{\mathsf{T}} X$ and X is a $\Sigma^{0,A}_3$ singleton. Then $X \leq_{\mathsf{LR}} Y$ implies $X' \leq_{\mathsf{T}} A \oplus Y'$.

Cor. $X^{(\alpha)} \leq_{\mathsf{LR}} Y$ implies $X^{(\alpha+1)} \leq_{\mathsf{T}} X \oplus Y'$.

Cor. $0^{(\alpha)} \leq_{\mathsf{LR}} Y$ implies $0^{(\alpha+1)} \leq_{\mathsf{T}} Y'$.

Lemma (Simpson 2009)**.**
If S is Σ^0_3 then

$$\begin{aligned} S^{\mathsf{LR}} &= \text{the LR-upward closure of } S \\ &= \{Y \mid (\exists X \in S)\,(X \leq_{\mathsf{LR}} Y)\} \end{aligned}$$

is again Σ^0_3.

Cor. For each recursive ordinal α, the set

$$\mathsf{B}_\alpha = \{Y \mid 0^{(\alpha)} \leq_{\mathsf{LR}} Y\}$$

is Σ^0_3.

Degrees of unsolvability:

A *mass problem* is a set of Turing oracles, $P \subseteq \{0,1\}^{\mathbb{N}}$.

We say that P *is weakly reducible to* Q, written $P \leq_{\mathrm{w}} Q$, if for each $Y \in Q$ there exists $X \in P$ such that $X \leq_{\mathrm{T}} Y$.

The *weak degree of* P, written $\deg_{\mathrm{w}}(P)$, is the equivalence class of P under mutual weak reducibility.

The weak degrees form a lattice, $\mathcal{D}_{\mathrm{w}}$.

Idea: P is a "problem". The elements of P are the "solutions" of P. P is "reducible" to Q if each "solution" of Q can be used as a Turing oracle to find some "solution" of P. Two "problems" are equivalent if each is "reducible" to the other.

Remark. Muchnik introduced $\mathcal{D}_{\mathrm{w}}$ in 1963 in order to rigorously explicate Kolmogorov's informal 1932 interpretation of intuitionism as a "calculus of problems."

A sublattice of $\mathcal{D}_{\mathsf{w}}$:

In recent years I have been studying a sublattice of $\mathcal{D}_{\mathsf{w}}$ which I call $\mathcal{E}_{\mathsf{w}}$. Namely, $\mathcal{E}_{\mathsf{w}}$ is the lattice of weak degrees of mass problems associated with nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$. It turns out that $\mathcal{E}_{\mathsf{w}}$ is extremely useful for classifying unsolvable mathematical problems. In particular, $\mathcal{E}_{\mathsf{w}}$ contains many, specific, natural degrees of unsolvability which are of interest from the viewpoint of foundations of mathematics.

Let $\mathbf{1}$ and $\mathbf{0}$ denote the top and bottom degrees in $\mathcal{E}_{\mathsf{w}}$. A handy technical lemma is:

Lemma (Simpson 2004)**.**
If $\mathbf{s} = \deg_{\mathsf{w}}(S)$ where S is Σ^0_3,
then $\inf(\mathbf{s}, \mathbf{1})$ belongs to $\mathcal{E}_{\mathsf{w}}$.

This applies to measure-theoretic regularity, because for each recursive ordinal α, the mass problem

$\{Y \mid 0^{(\alpha)} \leq_{\mathrm{LR}} Y\} = \{Y \mid$ every $\Sigma^0_{\alpha+2}$ set includes a $\Sigma^{0,Y}_2$ set of the same measure$\}$

is Σ^0_3.

Letting

$$\mathbf{b}_\alpha = \deg_{\mathrm{w}}(\{Y \mid 0^{(\alpha)} \leq_{\mathrm{LR}} Y\})$$

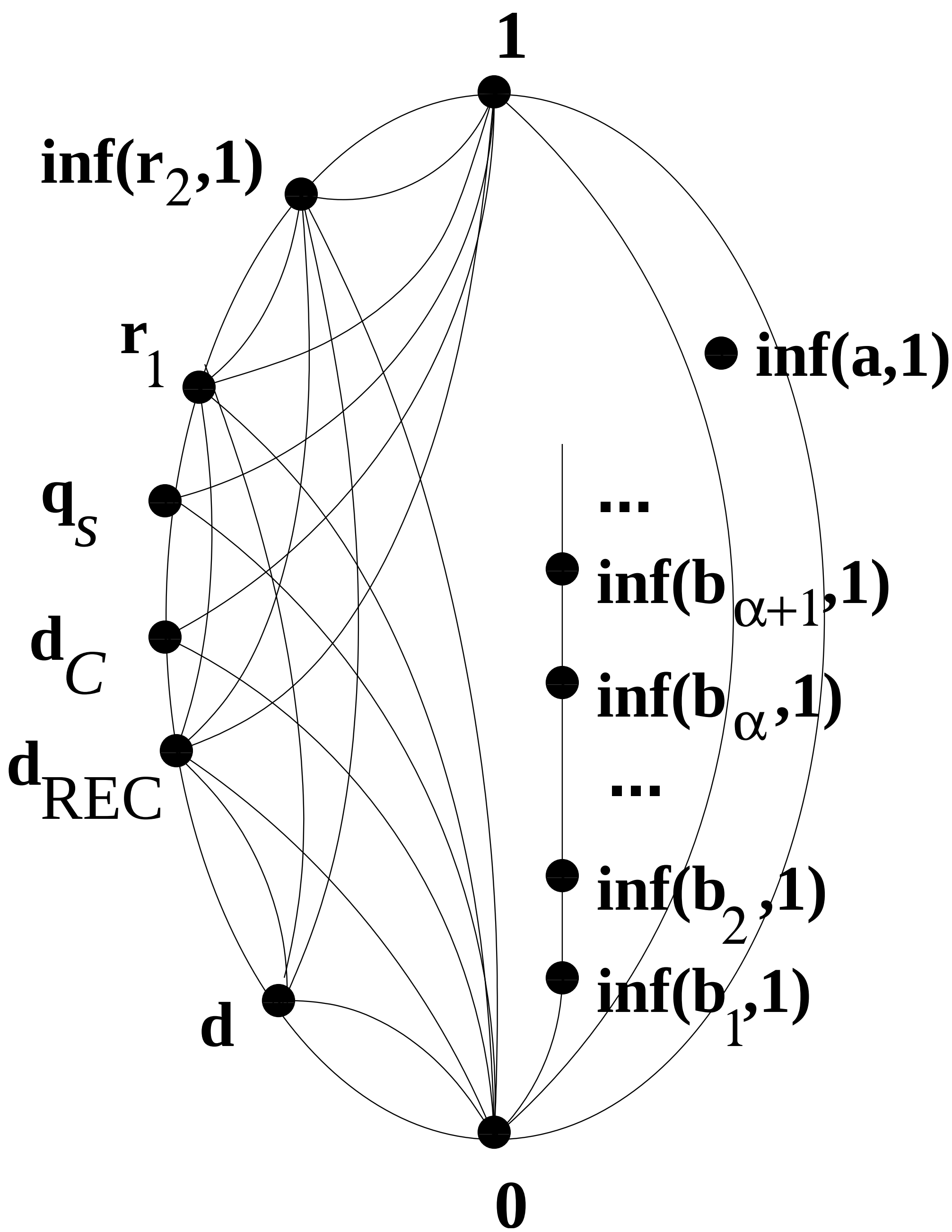
we have:

Theorem (Simpson 2009)**.** The weak degrees $\inf(\mathbf{b}_\alpha, \mathbf{1})$ belong to $\mathcal{E}_{\mathrm{w}}$. Moreover, for $\alpha > 0$ these degrees are distinct from one another and incomparable with previously known degrees in $\mathcal{E}_{\mathrm{w}}$ including

$$\mathbf{r}_1 = \deg_{\mathrm{w}}(\{Z \mid Z \text{ is Martin-Löf random}\})$$

and

$$\mathbf{d} = \deg_{\mathrm{w}}(\{f \mid f \text{ is diagonally nonrecursive}\}).$$



A picture of $\mathcal{E}_{\mathrm{w}}$. Here $\mathbf{a}$ = any r.e. degree, $\mathbf{b}$ = measure-theoretic regularity, $\mathbf{r}$ = randomness, $\mathbf{q}$ = Hausdorff dimension, $\mathbf{d}$ = diagonal nonrecursiveness.

Applications to reverse mathematics:

Applications to reverse mathematics follow, because many specific degrees in $\mathcal{D}_{\mathsf{w}}$ and $\mathcal{E}_{\mathsf{w}}$ are correlated to specific subsystems of second-order arithmetic. In particular $\mathsf{ACA}_0 \approx \mathbf{0}'$, $\mathsf{WKL}_0 \approx \mathbf{1}$, $\mathsf{WWKL}_0 \approx \mathbf{r}_1$.

Recall that ACA_0 and WKL_0 are members of the "Big Five." WWKL_0 is a weaker system which has been highly relevant for the reverse mathematics of measure theory.

Definition. Let M be an ω-model of RCA_0. Let S be a set in Euclidean space.

1. S is M-*Borel* if S is $\Sigma^{0,X}_\alpha$ for some $X \in M$ and $\alpha < \omega_1^X$.
2. S is M-F_σ if S is $\Sigma^{0,Y}_2$ for some $Y \in M$.
3. M is an MTR-*model* if every M-Borel set includes an M-F_σ set of the same measure.

Note: MTR = "measure-theoretic regularity."

Lemma. M is an MTR-model if and only if $(\forall X \in M)\,(\forall \alpha < \omega_1^X)\,(\exists Y \in M)\,(X^{(\alpha)} \leq_{\mathrm{LR}} Y)$.

Cor. If $M \models \mathsf{ATR}_0$ then M is an MTR-model.

Using our new results on LR-reducibility, we can build interesting MTR-models.

Theorem (Simpson 2009)**.** We can find MTR-models M_1, M_2, M_3, M_4 satisfying $\mathsf{RCA}_0 + \neg\,\mathsf{WWKL}_0$ and $\mathsf{WWKL}_0 + \neg\,\mathsf{WKL}_0$ and $\mathsf{WKL}_0 + \neg\,\mathsf{ACA}_0$ and $\mathsf{ACA}_0 + \neg\,\mathsf{ATR}_0$ respectively.

Moreover, these models can be made to satisfy measure-theoretic regularity at all levels of the Borel hierarchy corresponding to countable well-orderings with a small amount of transfinite induction.

Tentative conclusion:

The reverse mathematics of measure-theoretic regularity appears to be somewhat orthogonal to the Gödel hierarchy including the "Big Five" subsystems of second-order arithmetic.

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