Cone avoidance and randomness preservation

Stephen G. Simpson
Pennsylvania State University
http://www.math.psu.edu/simpson/
simpson@math.psu.edu

CCR 2014

Institute for Mathematical Sciences

National University of Singapore

June 9–13, 2014

Basis theorems.

A basis theorem is a theorem of the form:

> For any nonempty effectively closed set in Euclidean space, at least one member of the set is "close to being computable".

Some well known basis theorems are:

- the Low Basis Theorem,
- the R.E. Basis Theorem,
- the Hyperimmune-Free Basis Theorem,
- the Cone Avoidance Basis Theorem,
- the Randomness Preservation Basis Thm.

Less well known is a basis theorem of Higuchi/Hudelson/Simpson/Yokoyama on preservation of partial randomness.

Basis theorems are important for applications in the foundations of mathematics: models of arithmetic, Scott sets, ω-models of WKL_0 and WWKL_0, reverse mathematics, etc.

We discuss the possibilities for combining these basis theorems.

Three basis theorems.

Let $\leq_{\mathrm{T}}$ denote Turing reducibility.

Let $'$ denote the Turing jump operator.

<u>The Low Basis Theorem</u>:

For any nonempty effectively closed set Q, there exists $Z \in Q$ such that $Z' \leq_{\mathrm{T}} 0'$.

<u>The R.E. Basis Theorem</u>:

For any nonempty effectively closed set Q, there exists $Z \in Q$ such that Z is of recursively enumerable Turing degree.

We say that Z is *hyperimmune-free* if ($\forall$ functions $f \leq_{\mathrm{T}} Z$) ($\exists$ recursive function g) $\forall n\,(f(n) < g(n))$.

<u>The Hyperimmune-Free Basis Theorem</u>:

For any nonempty effectively closed set Q, $(\exists Z \in Q)$ (Z is hyperimmune-free).

These three basis theorems are due to Jockusch/Soare 1972.

Can we combine these basis theorems?

No. The Jockusch/Soare basis theorems are known to be "pairwise incompatible."

1. The Arslanov Completeness Criterion provides a nonempty effectively closed Q such that for all r.e. sets A,
   if $(\exists Z \in Q)\,(Z \leq_{\mathrm{T}} A)$ then $0' \leq_{\mathrm{T}} A$.

   Therefore, the Low Basis Theorem and the R.E. Basis Theorem cannot be combined into one basis theorem.

2. It is known that for hyperimmune-free Z one cannot have $0 <_{\mathrm{T}} Z \leq_{\mathrm{T}} 0'$.

   Therefore, the Hyperimmune-Free Basis Theorem cannot be combined with the Low Basis Theorem or with the R.E. Basis Theorem.

Two more basis theorems.

<u>The Cone Avoidance Basis Theorem</u>:

For any nonempty effectively closed set Q, if $A \not\leq_{\mathrm{T}} 0$ then $(\exists Z \in Q)\,(A \not\leq_{\mathrm{T}} Z)$.

More generally, if $\forall i\,(A_i \not\leq_{\mathrm{T}} 0)$ then $(\exists Z \in Q)\,\forall i\,(A_i \not\leq_{\mathrm{T}} Z)$.

Gandy/Kreisel/Tait, 1960.

Let $\mathsf{MLR} = \{X \mid X$ is Martin-Löf random$\}$.
Let $\mathsf{MLR}^Z = \{X \mid X$ is Martin-Löf random relative to $Z\}$.

<u>The Randomness Preservation Basis Theorem</u>:

For any nonempty effectively closed set Q, if $X \in \mathsf{MLR}$ then $(\exists Z \in Q)\,(X \in \mathsf{MLR}^Z)$.

Reimann/Slaman, 2005,

Downey/Hirschfeldt/Miller/Nies, 2005,

Simpson/Yokoyama, 2011.

More combinations of basis theorems?

It is known that Cone Avoidance can be combined with the Low Basis Theorem, or with the Hyperimmune-free Basis Theorem, but not with the R.E. Basis Theorem. (See for instance Downey/Hirschfeldt §2.19.3.)

Also, Randomness Preservation cannot be combined with the Low or the R.E. or the Hyperimmune-Free Basis Theorem.

Specifically, let $\Omega \in \mathsf{MLR}$ be such that $\Omega \equiv_{\mathsf{T}} 0'$. It is known that such reals exist (Chaitin, Kučera/Gács). We then have:

1. Any $Z \leq_{\mathsf{T}} 0'$ such that $\Omega \in \mathsf{MLR}^Z$ is K-trivial, hence not PA-complete. (See Chapter 11 of Downey/Hirschfeldt 2010 or Chapter 5 of Nies 2009.)

2. Any hyperimmune-free Z such that $\Omega \in \mathsf{MLR}^Z$ is recursive. (See Theorem 8.1.18 of Nies 2009.)

Combining basis theorems.

	Low	R.E.	H.I.F.	C.A.	R.P.
Low	1	0	0	1	0
R.E.	0	1	0	0	0
H.I.Free	0	0	1	1	0
Cone Av.	1	0	1	1	**???**
Rand. Pres.	0	0	0	**???**	1

Remaining question: Can Cone Avoidance be combined with Randomness Preservation?

The answer to this question involves LR-reducibility.

Define $A \leq_{\mathsf{LR}} B \iff \mathsf{MLR}^B \subseteq \mathsf{MLR}^A$. Clearly $A \leq_{\mathsf{T}} B$ implies $A \leq_{\mathsf{LR}} B$, and it is known that $A \leq_{\mathsf{LR}} 0$ implies $A' \leq_{\mathsf{T}} 0'$. A major theorem of Nies is that $A \leq_{\mathsf{LR}} 0 \iff A$ is K-trivial. See Nies 2009 or Downey/Hirschfeldt 2010.

A theorem which combines Cone Avoidance and Randomness Preservation:

Theorem 1 (Simpson/Stephan, 2013)**.**
For any nonempty effectively closed set Q, if $X \in \mathsf{MLR}$ and $\forall i\,(A_i \not\leq_{\mathsf{LR}} 0$ or $A_i \not\leq_{\mathsf{T}} X)$, then $(\exists Z \in Q)\,(X \in \mathsf{MLR}^Z$ and $\forall i\,(A_i \not\leq_{\mathsf{T}} Z))$.

On the other hand, let $\Omega \in \mathsf{MLR}$ be such that $\Omega \equiv_{\mathsf{T}} 0'$. It is well known that such reals exist (Chaitin, Kučera/Gács).

Theorem 2 (Simpson/Stephan, 2013)**.**
$\exists$ nonempty effectively closed set Q such that $(\forall A \leq_{\mathsf{LR}} 0)\,(\forall Z \in Q)\,(\Omega \in \mathsf{MLR}^Z \Rightarrow A \leq_{\mathsf{T}} Z)$.

Summary of Theorems 1 and 2:

Cone Avoidance is "almost compatible" with Randomness Preservation.

The only obstacle to full compatibility is the existence of non-computable K-trivial cones, i.e., $A \leq_{\mathsf{LR}} 0$ and $A \not\leq_{\mathsf{T}} 0$.

Proofs of Theorems 1 and 2.

To prove Theorem 1, we use
the Cone Avoidance Basis Theorem,
relativized to X.

To prove Theorem 2, we use
K = prefix-free Kolmogorov complexity.

(1) If $\Omega \in \mathsf{MLR}^Z$ then $|\mathsf{K}(n) - \mathsf{K}^Z(n)| \leq O(1)$
for infinitely many n. (Miller 2010.)

(2) If $\Omega \in \mathsf{MLR}^Z$ and Z is PA-complete,
then there exist a Z-recursive function F
and an infinite Z-recursive set A such that
$|\mathsf{K}(n) - F(n)| \leq O(1)$ for all $n \in A$.

(3) Let C = plain Kolmogorov complexity.
Chaitin 1976 proved: every C-trivial real is
computable. Using F and A as in (2), we
similarly prove: every K-trivial real is $\leq_{\mathsf{T}} Z$.

For details, see Simpson/Stephan 2013.

An application.

Recall that WKL_0 is a subsystem of Z_2 which is good for the reverse mathematics of compactness (Heine-Borel, Arzela-Ascoli, Hahn-Banach, fixed points, prime ideals, . . .).

And, WWKL_0 is a subsystem of WKL_0 which is good for the reverse mathematics of measure theory (countable additivity, Monotone and Dominated Convergence theorems, Vitali Covering Lemma, . . .).

Let M be a countable ω-model of WWKL_0.

By Simpson/Yokoyama 2011, we get a countable ω-model $M_1 \supseteq M$ of WKL_0 such that $C \cap M \neq \emptyset$ for every M_1-coded closed set C of positive measure.

Call this a good extension of M.

As an application of Theorem 1, we get two good extensions $M_1, M_2 \supseteq M$ such that $M = M_1 \cap M_2$.

Partial randomness.

Let $f : \{0,1\}^* \to [0,\infty)$ be computable.

For $S \subseteq \{0,1\}^*$ let $\mathrm{wt}_f(S) = \sum_{\sigma \in S} 2^{-f(\sigma)}$,

$\mathrm{pwt}_f(S) = \sup\{\mathrm{wt}_f(P) \mid P \subseteq S \text{ prefix-free}\}$,

and $[\![S]\!] = \{X \in \{0,1\}^{\mathbb{N}} \mid \exists n\,(X{\upharpoonright}n \in S)\}$.

We say that X is strongly f-random if

$X \notin \bigcap_{i=0}^{\infty} [\![S_i]\!]$ for all uniformly r.e. $S_i \subseteq \{0,1\}^*$

such that $\forall i\,(\mathrm{pwt}_f(S_i) \leq 2^{-i})$.

Martin-Löf randomness is the special case $f(\sigma) = |\sigma|$. In this case $\mathrm{pwt}_f(S) = \lambda([\![S]\!])$ where λ is the fair coin measure on $\{0,1\}^{\mathbb{N}}$.

Theorem (Partial Randomness Preservation, Higuchi/Hudelson/Simpson/Yokoyama 2011)**.**

For any nonempty effectively closed set Q, if X is strongly f-random then $(\exists Z \in Q)$ (X is strongly f-random relative to Z).

More generally, if $\forall i\,(X_i$ is strongly f_i-random$)$ then $(\exists Z \in Q)\,\forall i\,(X_i$ is strongly f_i-random relative to $Z)$.

Problem: To what extent can we combine the *Partial* Randomness Preservation Basis Theorem with cone avoidance?

Theorem 3 (implicit in H/H/S/Y 2011)**.** For any nonempty effectively closed set Q, if $\forall i\,(A_i \not\leq_{\mathsf{LR}} 0$ and X_i is strongly f_i-random), then $(\exists Z \in Q)\,\forall i\,(A_i \not\leq_{\mathsf{LR}} Z$ and X_i is strongly f_i-random relative to Z).

On the other hand, because of Theorem 2, we cannot *always* replace $\leq_{\mathsf{LR}}$ by $\leq_{\mathsf{T}}$.

Can we *sometimes* replace $\leq_{\mathsf{LR}}$ by $\leq_{\mathsf{T}}$?

A typical question:

Define X to be <u>strongly half-random</u> $\iff$ X is strongly f-random where $f(\sigma) = |\sigma|/2$.

Let Q be nonempty and effectively closed.

If $A \not\leq_{\mathsf{T}} 0$ and X is strongly half-random, does there exist $Z \in Q$ such that $A \not\leq_{\mathsf{T}} Z$ and X is strongly half-random relative to Z?

Recent literature.

Jan Reimann and Theodore A. Slaman,
Measures and their random reals, 15 pages, 2008,
Transactions of the American Mathematical Society,
accepted for publication.

Rod Downey, Denis R. Hirschfeldt, Joseph S. Miller,
and André Nies, Relativizing Chaitin's halting
probability, Journal of Mathematical Logic, 5, 2005,
167–192.

Joseph S. Miller, The K-degrees, low-for-K degrees,
and weakly low-for-K sets, Notre Dame Journal of
Formal Logic, 50, 2010, 381–391.

Stephen G. Simpson and Keita Yokoyama,
A nonstandard counterpart of WWKL, Notre Dame
Journal of Formal Logic, 52, 2011, 229–243.

Vasco Brattka, Joseph S. Miller, and André Nies,
Randomness and differentiability, 39 pages, 2011,
Transactions of the American Mathematical Society,
accepted for publication.

Kojiro Higuchi, W. M. Phillip Hudelson, Stephen G.
Simpson, and Keita Yokoyama, Propagation of partial
randomness, Annals of Pure and Applied Logic, 165,
2014, 742–758.

Stephen G. Simpson and Frank Stephan,
Cone avoidance and randomness preservation,
22 pages, 2013, Annals of Pure and Applied Logic,
conditionally accepted for publication.

More on partial randomness.

Under mild hypotheses on f, Hudelson proved the existence of a strong f-random which does not compute any (strong) g-random, if g grows significantly faster than f.

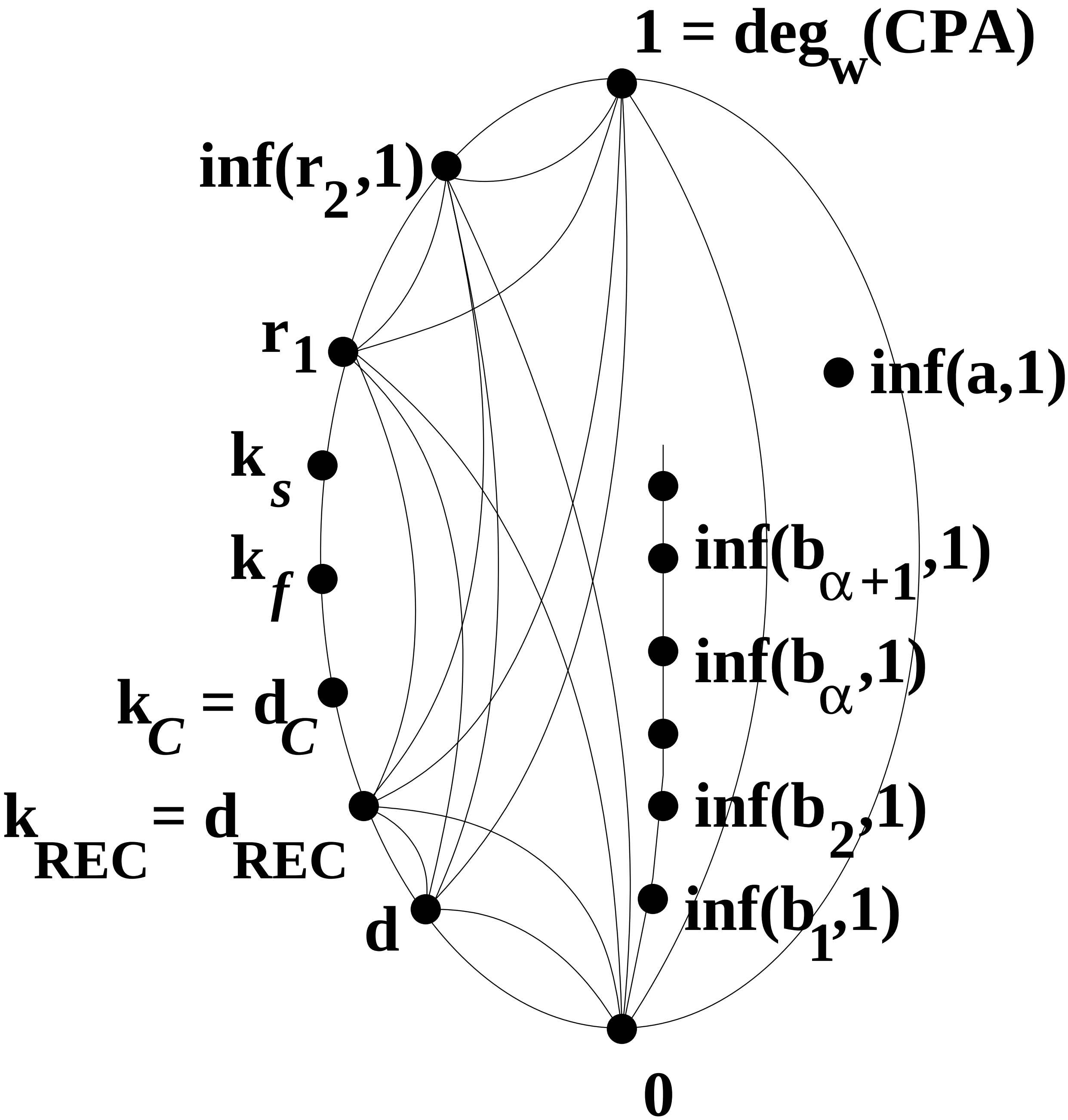
Theorem (Hudelson). Let f and g be computable, convex, unbounded, and length-invariant such that $f(\sigma) + 2\log_2 f(\sigma) \le g(\sigma)$ for all σ. Then $\exists X\,(X$ is strongly f-random and $(\forall Y \le_{\mathrm{T}} X)\,(Y$ is not g-random$))$.

This generalizes results of Miller and Greenberg/Miller.

Example. We can find an X such that $\mathrm{K}(X{\upharpoonright}n) \ge^{+} \sqrt{n}$ and there is no $Y \le_{\mathrm{T}} X$ such that $\mathrm{K}(Y{\upharpoonright}n) \ge^{+} \sqrt{n} + \log_2 n$.

Reference. W. M. Phillip Hudelson, Mass problems and initial segment complexity, Journal of Symbolic Logic, 79, 2014, 20–44.

Note. Hudelson's theorem provides many natural examples of Muchnik degrees of mass problems associated with nonempty Π^0_1 subsets of $\{0,1\}^{\mathbb{N}}$.



Partial randomness and μ-randomness.

Several authors (Levin/Gács, Day/Miller, Reimann/Slaman, Day/Reimann, ...) have defined what it means for $X \in \{0,1\}^{\mathbb{N}}$ to be μ-random where μ is an arbitrary Borel probability measure on $\{0,1\}^{\mathbb{N}}$.

If $\mu = \lambda =$ the fair coin measure on $\{0,1\}^{\mathbb{N}}$, then μ-randomness = Martin-Löf randomness.

In general, μ need not be computable.

From now on, let $f : \{0,1\}^* \to [0,\infty)$ be computable and convex, i.e., $\mathrm{wt}_f(\sigma) \leq \mathrm{wt}_f(\sigma^\frown\langle 0\rangle) + \mathrm{wt}_f(\sigma^\frown\langle 1\rangle)$ for all σ. This is a mild assumption.

We can characterize strong f-randomness in terms of μ-randomness:

Effective Capacitability Theorem (Reimann, Kjos-Hanssen, Simpson/Stephan)**.** X is strongly f-random $\Longleftrightarrow$ $\exists\mu\,(X \text{ is } \mu\text{-random} \wedge \exists c \forall\sigma\,(\mu([\![\sigma]\!]) \leq 2^{c-f(\sigma)}))$.

A <u>product theorem</u> for $\mu \times \nu$-randomness:

Theorem. $X \oplus Y$ is $\mu \times \nu$-random $\iff X$ is μ-random and Y is ν-random relative to X, μ.

Combine with Effective Capacitability:

Theorem 4 (Simpson/Stephan 2013)**.**
If X is strongly f-random, and
if Y is Martin-Löf random relative to X,
then X is strongly f-random relative to Y.

Theorem 4 resembles an older result:

Theorem (H/H/S/Y 2011)**.**
If X is strongly f-random and $\leq_{\mathsf{T}} Y$
where Y is Martin-Löf random relative to Z,
then X is strongly f-random relative to Z.

However, we do not know how to deduce Theorem 4 from H/H/S/Y or vice versa.

Jason Rute has used Effective Capacitability to prove an <u>Ample Excess Lemma</u> for strong f-randomness:

Theorem (Rute)**.** If X is strongly f-random, then $\displaystyle\sum_{n=0}^{\infty} 2^{f(X\restriction n)-\mathsf{K}(X\restriction n)} < \infty$.

Complexity and autocomplexity.

Definition (Kjos-Hanssen/Merkle/Stephan).
$X \in \{0,1\}^{\mathbb{N}}$ is complex (autocomplex)
if there exists an unbounded $h : \mathbb{N} \to \mathbb{N}$
such that $\mathsf{K}(X{\upharpoonright}n) \geq h(n)$ for all n, and
h is computable (computable from X).

Theorem (K-H/M/S 2006).
X is complex (autocomplex) $\Longleftrightarrow$
there exists a DNR function $\leq_{\mathsf{tt}} X$ ($\leq_{\mathsf{T}} X$).

Theorem (H/H/S/Y 2011).
X is autocomplex (complex) $\Longleftrightarrow$
X is strongly f-random for some
computable (computable length-invariant)
f such that $\{f(X{\upharpoonright}n) \mid n \in \mathbb{N}\}$ is unbounded.

Theorem (Reimann/Slaman, Simpson/Stephan).

1. X is autocomplex relative to some oracle
$\Longleftrightarrow \exists \mu\,(X$ is μ-random $\wedge\, \mu(\{X\}) = 0)$,
$\Longleftrightarrow X$ is non-computable.

2. X is complex relative to some oracle
$\Longleftrightarrow \exists \mu\,(X$ is μ-random $\wedge\, \forall Y\, (\mu(\{Y\}) = 0))$.

Literature.

Jan Reimann and Theodore A. Slaman,
Measures and their random reals, 15 pages, 2008,
Transactions of the American Mathematical Society,
accepted for publication.

Bjørn Kjos-Hanssen, Wolfgang Merkle, and Frank Stephan, Kolmogorov complexity and the recursion theorem, Transactions of the American Mathematical Society, 363, 2011, 5465–5480.

Adam R. Day and Joseph S. Miller,
Randomness for non-computable measures,
Transactions of the American Mathematical Society,
365, 2013, 3575–3591.

Kojiro Higuchi, W. M. Phillip Hudelson, Stephen G. Simpson, and Keita Yokoyama, Propagation of partial randomness, Annals of Pure and Applied Logic, 165, 2014, 742–758.

Stephen G. Simpson and Frank Stephan,
Cone avoidance and randomness preservation,
22 pages, 2013, Annals of Pure and Applied Logic,
conditionally accepted for publication.

André Nies, Computability and Randomness,
Oxford University Press, 2009, XV + 433 pages.

Rodney G. Downey and Denis R. Hirschfeldt,
Algorithmic Randomness and Complexity,
Springer, 2010, XXVIII + 855 pages.

Thank you for your attention!