Symbolic Dynamics:

Entropy = Dimension = Complexity

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CCA-CCR Conference

University of Cape Town

January 31 – February 4, 2011

Symbolic dynamics.

Let G be $(\mathbb{N}^d, +)$ or $(\mathbb{Z}^d, +)$ where $d \geq 1$.

Let A be a finite set of symbols.

We endow A with the discrete topology and A^G with the product topology.

The *shift action* of G on A^G is given by $(S^g x)(h) = x(g+h)$ for $g, h \in G$ and $x \in A^G$.

A *subshift* is a nonempty set $X \subseteq A^G$ which is topologically closed and *shift-invariant*, i.e., $x \in X$ implies $S^g x \in X$ for all $g \in G$.

Symbolic dynamics is the study of subshifts.

If $X \subseteq A^G$ and $Y \subseteq B^G$ are subshifts, a *shift morphism* from X to Y is a continuous mapping $\Phi : X \to Y$ such that $\Phi(S^g x) = S^g \Phi(x)$ for all $x \in X$ and $g \in G$.

By compactness, any shift morphism Φ is given by a *block code*, i.e., a finite mapping $\phi : A^F \to B$ where F is a finite subset of G and $\Phi(x)(g) = \phi(S^g x \restriction F)$ for all $x \in X$ and $g \in G$.

Some new (!?!) results on subshifts:

Let d be a positive integer, let A be a finite set of symbols, and let X be a nonempty subset of A^G where G is $\mathbb{N}^d$ or $\mathbb{Z}^d$.

The *Hausdorff dimension*, $\dim(X)$, and the *effective Hausdorff dimension*, $\mathrm{effdim}(X)$, are defined as usual with respect to the standard metric $\rho(x,y) = 2^{-|F_n|}$ where n is as large as possible such that $x{\upharpoonright}F_n = y{\upharpoonright}F_n$.

Here F_n is $\{1,\ldots,n\}^d$ if $G = \mathbb{N}^d$,
or $\{-n,\ldots,n\}^d$ if $G = \mathbb{Z}^d$.

We first state some old results.

1. $\mathrm{effdim}(X) = \sup\limits_{x \in X} \mathrm{effdim}(x)$.

2. $\mathrm{effdim}(x) = \liminf\limits_{n\to\infty} \dfrac{\mathsf{K}(x{\upharpoonright}F_n)}{|F_n|}$.

3. $\mathrm{effdim}(X) = \dim(X)$
   provided X is <u>effectively closed</u>, i.e., Π^0_1.

Here K denotes Kolmogorov complexity.

Theorem (Simpson 2010)**.** Assume that X is a subshift, i.e., X is <u>closed</u> and <u>shift-invariant</u>. Then

$$\mathsf{effdim}(X) = \dim(X) = \mathsf{ent}(X).$$

Moreover

$$\dim(X) \geq \limsup_{n\to\infty} \frac{\mathsf{K}(x{\restriction}F_n)}{|F_n|} \quad \text{for all } x \in X,$$

and

$$\dim(X) = \lim_{n\to\infty} \frac{\mathsf{K}(x{\restriction}F_n)}{|F_n|} \quad \text{for many } x \in X.$$

Remark. Here $\mathsf{ent}(X)$ denotes *entropy*,

$$\mathsf{ent}(X) = \lim_{n\to\infty} \frac{\log_2 |\{x{\restriction}F_n \mid x \in X\}|}{|F_n|}.$$

This is known to be a conjugacy invariant.

Note. In the above theorem, there is no finiteness or computability hypothesis on the subshift X. Moreover, X can be a G-subshift where G is $\mathbb{N}^d$ or $\mathbb{Z}^d$ for any positive integer d.

Remark. The proof of the theorem involves ergodic theory (Shannon/McMillan/Breiman, the Variational Principle, etc.) plus a combinatorial argument which is similar to the proof of the Vitali Covering Lemma.

Remark. The above theorem seems so fundamental that it must have been noticed long ago. Nevertheless, I have not been able to find it in the literature. So far as I can tell, everything in the theorem is new, except the following result of Furstenberg 1967:

$$\text{dim}(X) = \text{ent}(X) \text{ provided } G = \mathbb{N}.$$

The proof of this special case is much easier.

Remark. The above theorem is an outcome of my discussions at Penn State during February–April 2010 with many people including John Clemens, Mike Hochman, Dan Mauldin, Jan Reimann, and Sasha Shen.

Degrees of unsolvability (Muchnik).

Let X be any set of reals. We view X as a *mass problem*, viz., the problem of "finding" <u>some</u> $x \in X$.

In order to interpret "finding," we use Turing's concept of computability.

Accordingly, we say that X is *algorithmically solvable* if X contains <u>some</u> computable real, or in other words, $X \cap \text{REC} \neq \emptyset$.

Similarly, we say that X *is algorithmically reducible to* Y if <u>each</u> $y \in Y$ can be used as a Turing oracle to compute <u>some</u> $x \in X$.

The *degree of unsolvability* of X, deg(X), is the equivalence class of X under mutual algorithmic reducibility.

Reference:

Albert A. Muchnik, On strong and weak reducibilities of algorithmic problems, Sibirskii Matematicheskii Zhurnal, 4, 1963, 1328–1341, in Russian.

I have been applying recursion-theoretic concepts such as Muchnik degrees and Kolmogorov complexity to obtain new results in symbolic dynamics.

Muchnik degrees of subshifts.

A subshift X is *of finite type* if it is given by a finite set of excluded finite configurations:

$$X = \{x \in A^G \mid (\forall g \in G)\,(S^g x{\restriction}F \notin E)\}$$

where E and F are finite.

Recall that $\mathcal{E}_\mathsf{w}$ is the lattice of Muchnik degrees of nonempty Π^0_1 classes, in Cantor space (or in Euclidean space).

Recall also that $\mathcal{E}_\mathsf{w}$ includes many specific, natural degrees which are associated with foundationally interesting topics.

A picture of $\mathcal{E}_\mathsf{w}$ is on slides 10 and 12.

Theorem (Simpson 2007)**.** The Muchnik degrees in $\mathcal{E}_\mathsf{w}$ are precisely the Muchnik degrees of $\mathbb{Z}^2$-subshifts of finite type.

Proof. One direction is trivial, because subshifts of finite type may be viewed as Π^0_1 classes. My proof of the other direction uses tiling techniques which go back to Berger 1966, Robinson 1971, Myers 1974. Another proof, due to Durand/Romashchenko/Shen 2008, uses "self-replicating tile sets."

Corollary (Simpson 2007)**.** We can construct an infinite family of $\mathbb{Z}^2$-subshifts of finite type which are strongly independent with respect to shift morphisms, etc.

Proof. This follows from the existence of an infinite independent set of degrees in $\mathcal{E}_{\mathrm{w}}$. The existence of such degrees is proved by means of a priority argument.

Thus we have an application of recursion theory (tiling methods plus priority argument) to prove a result in symbolic dynamics which does not mention computability concepts.

A possibly interesting research program:

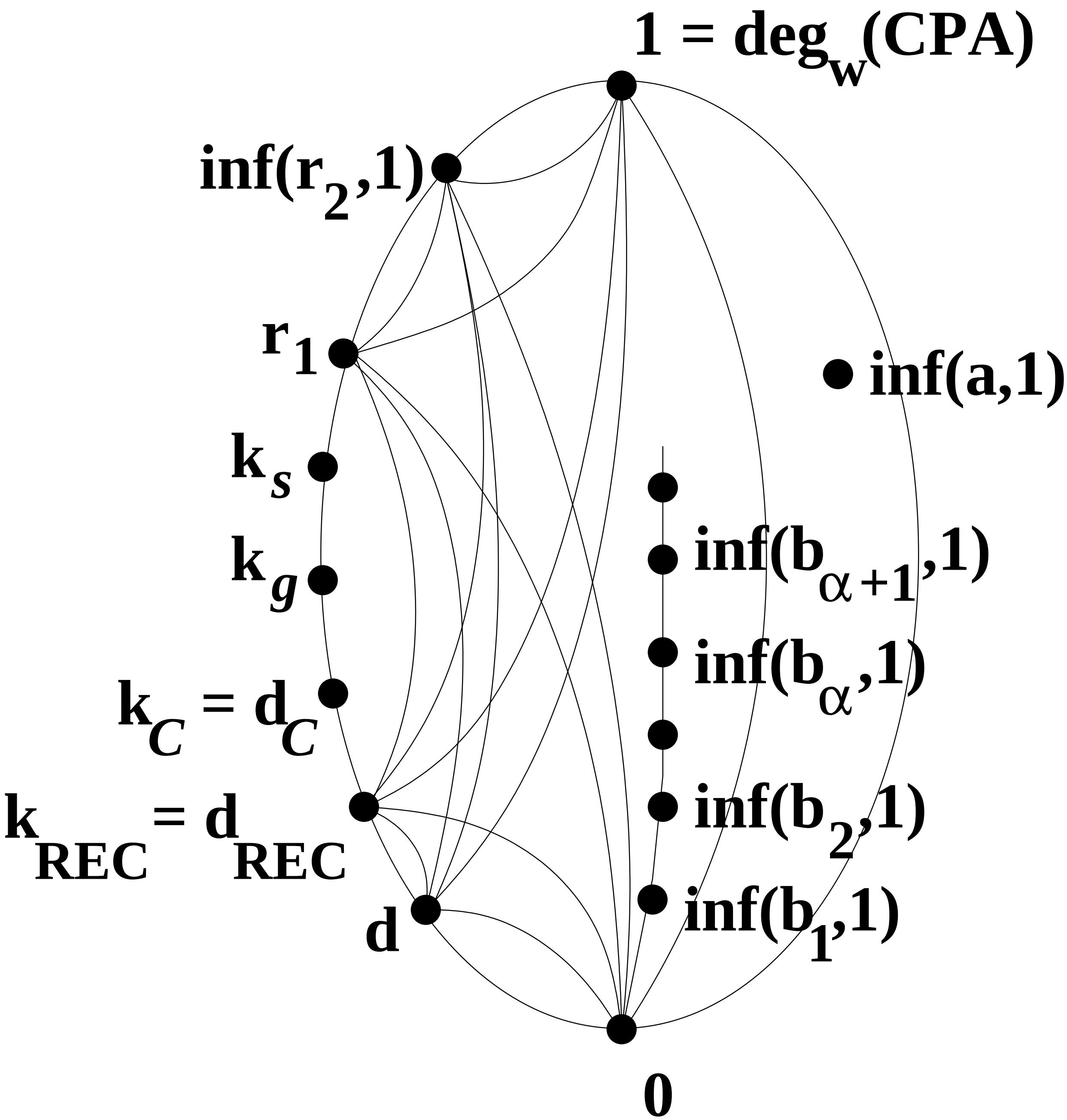
Given a subshift X, explore the relationship between the *dynamical properties* of X and the *degree of unsolvability* of X, i.e., its Muchnik degree, deg(X).

For example, the *entropy* of X is a well-known dynamical property which serves as an upper bound on the complexity of orbits. In particular ent$(X) > 0$ implies $(\exists x \in X)\,(x$ is not computable).

By contrast, the degree of unsolvability of X serves as a lower bound on the complexity of orbits. For instance, deg$(X) > \mathbf{0} \Longleftrightarrow$ $(\forall x \in X)\,(x$ is not computable).

Theorem (Hochman)**.** If X is of finite type and *minimal* (i.e., every orbit is dense), then deg$(X) = \mathbf{0}$.

More generally, the theorem holds for all Π^0_1 subshifts, not necessarily of finite type.



A picture of $\mathcal{E}_w$. Each black dot except $\inf(\mathbf{a}, \mathbf{1})$ represents a specific, natural degree in $\mathcal{E}_w$. We shall explain some of these degrees.

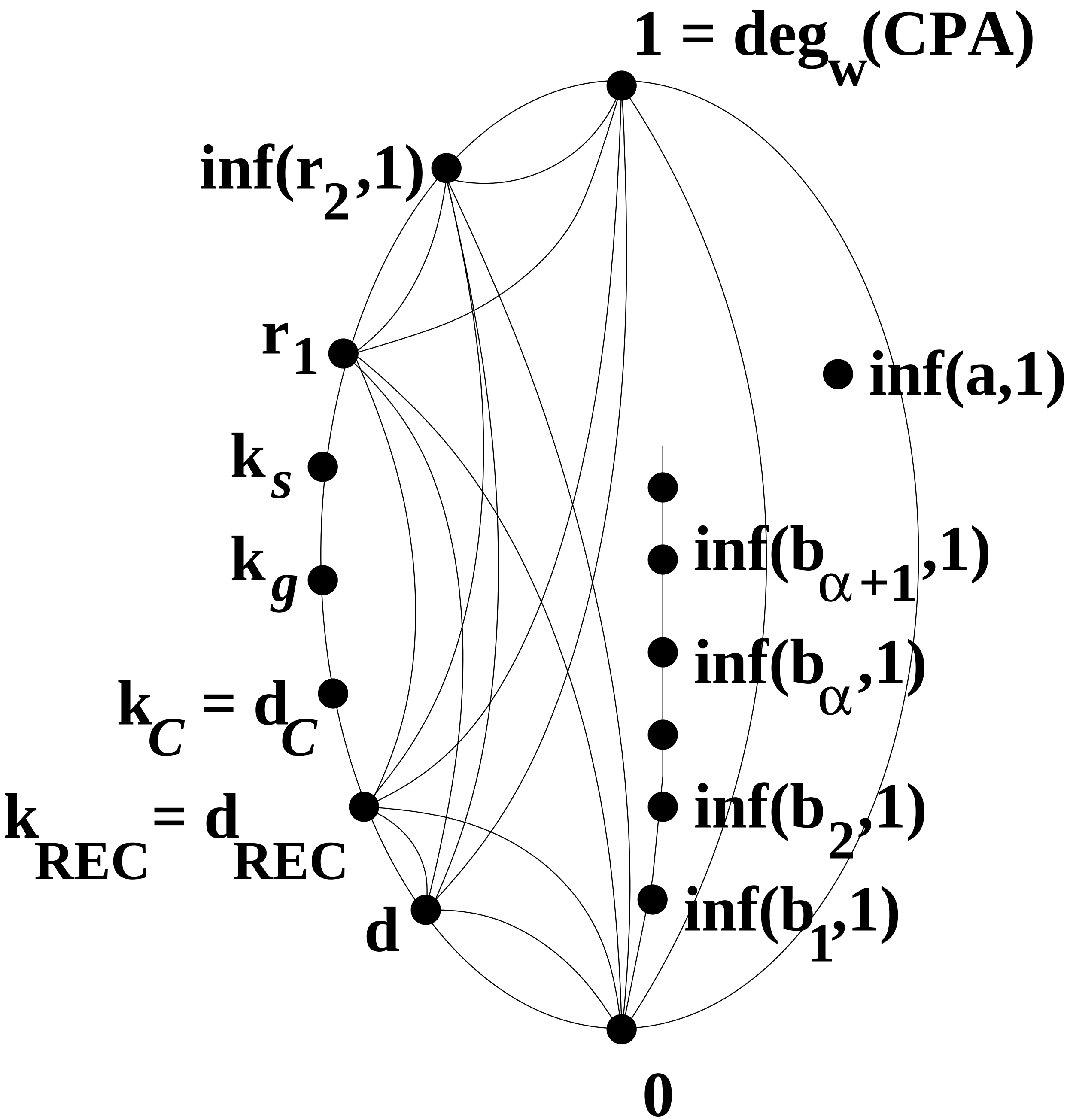
Two subshifts are said to be *conjugate* if they are topologically isomorphic, i.e., there is a shift isomorphism between them.

The basic problem of symbolic dynamics is: classify subshifts up to conjugacy invariance.

Muchnik degrees can help, because the Muchnik degree of a subshift is a conjugacy invariant. In particular, each degree in $\mathcal{E}_{\mathsf{w}}$ including $\mathbf{0}$, $\mathbf{1}$, $\mathbf{r}_1$, $\mathbf{d}$, $\mathbf{d}_{\mathsf{REC}}$, $\mathbf{d}_C$, $\mathbf{k}_s$, $\mathbf{k}_g$, $\inf(\mathbf{r}_2, \mathbf{1})$, $\inf(\mathbf{b}_\alpha, \mathbf{1})$, and even $\inf(\mathbf{a}, \mathbf{1})$ may be viewed as a conjugacy invariant for subshifts of finite type.

It is interesting to compare the Muchnik degree of a subshift X with other conjugacy invariants, e.g., the entropy of X.

Generally speaking, the Muchnik degree of X represents a lower bound on the complexity of the orbits, while the entropy of X is an upper bound on the complexity of these same orbits.



A picture of $\mathcal{E}_{\mathrm{w}}$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{r} =$ randomness, $\mathbf{b} =$ LR-reducibility, $\mathbf{k} =$ complexity, $\mathbf{d} =$ diagonal nonrecursiveness.

We now explain some degrees in $\mathcal{E}_{\mathsf{w}}$.

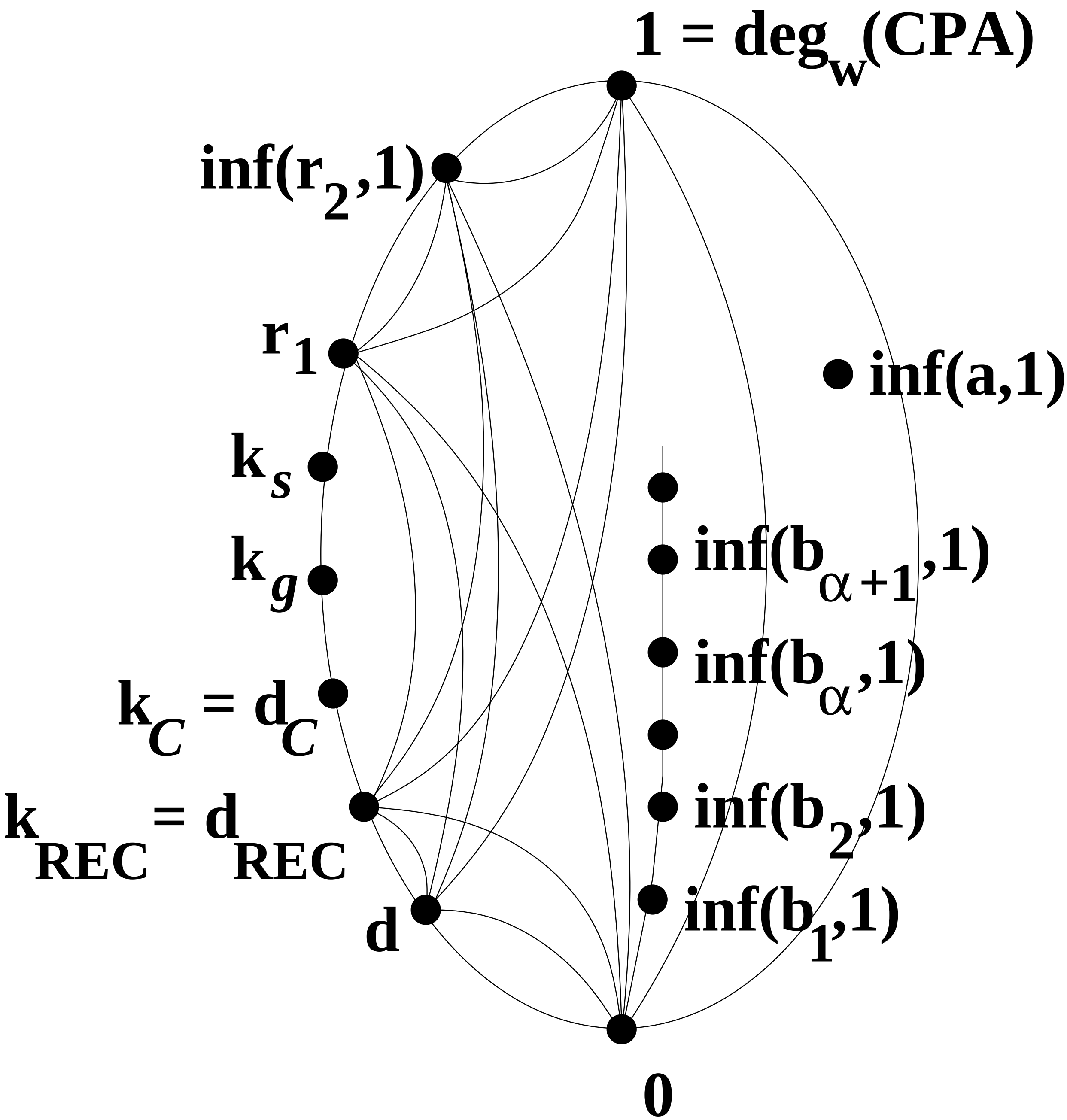
The top degree in $\mathcal{E}_{\mathsf{w}}$ is $\mathbf{1} = \deg(\mathsf{CPA})$ where CPA is the problem of finding a complete consistent theory which includes Peano arithmetic (or ZFC, etc.).

We also have $\inf(\mathbf{a},\mathbf{1}) \in \mathcal{E}_{\mathsf{w}}$ where $\mathbf{a}$ is any recursively enumerable Turing degree. Moreover, $\mathbf{a} < \mathbf{b}$ implies $\inf(\mathbf{a},\mathbf{1}) < \inf(\mathbf{b},\mathbf{1})$

We have $\mathbf{r}_1 \in \mathcal{E}_{\mathsf{w}}$ where $\mathbf{r}_1 = \deg(\mathsf{MLR})$, $\mathsf{MLR} = \{x \in 2^{\mathbb{N}} \mid x$ is Martin-Löf random$\}$).

We also have $\inf(\mathbf{r}_2,\mathbf{1}) \in \mathcal{E}_{\mathsf{w}}$ where $\mathbf{r}_2 = \deg(\{x \in 2^{\mathbb{N}} \mid x$ is 2-*random*$\}$), i.e., random relative to the halting problem.

Also $\mathbf{d} \in \mathcal{E}_{\mathsf{w}}$ where $\mathbf{d} =$ $\deg(\{f \mid f$ is *diagonally nonrecursive*$\}$), i.e., $\forall n\,(f(n) \neq \varphi_n(n))$.



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Let $\mathsf{REC} = \{g \in \mathbb{N}^{\mathbb{N}} \mid g$ is recursive$\}$.
Let C be any "nice" subclass of REC.
For instance $C = \mathsf{REC}$, or $C = \{g \in \mathsf{REC} \mid g$ is primitive recursive$\}$. We have $\mathbf{d}_C \in \mathcal{E}_{\mathsf{w}}$ where $\mathbf{d}_C = \deg(\{f \in \mathbb{N}^{\mathbb{N}} \mid f$ is diagonally nonrecursive and C-*bounded*$\})$,
i.e., $(\exists g \in C)\, \forall n\, (f(n) < g(n))$.

Also, $\mathbf{d}_C = \deg(\{x \in 2^{\mathbb{N}} \mid x$ is C-*complex*$\}$, i.e., $(\exists g \in C)\, \forall n\, (\mathsf{K}(x{\restriction}\{1, \ldots, g(n)\}) \geq n)\})$.
Moreover, $\mathbf{d}_{C'} < \mathbf{d}_C$ whenever C' contains a function which dominates all functions in C.

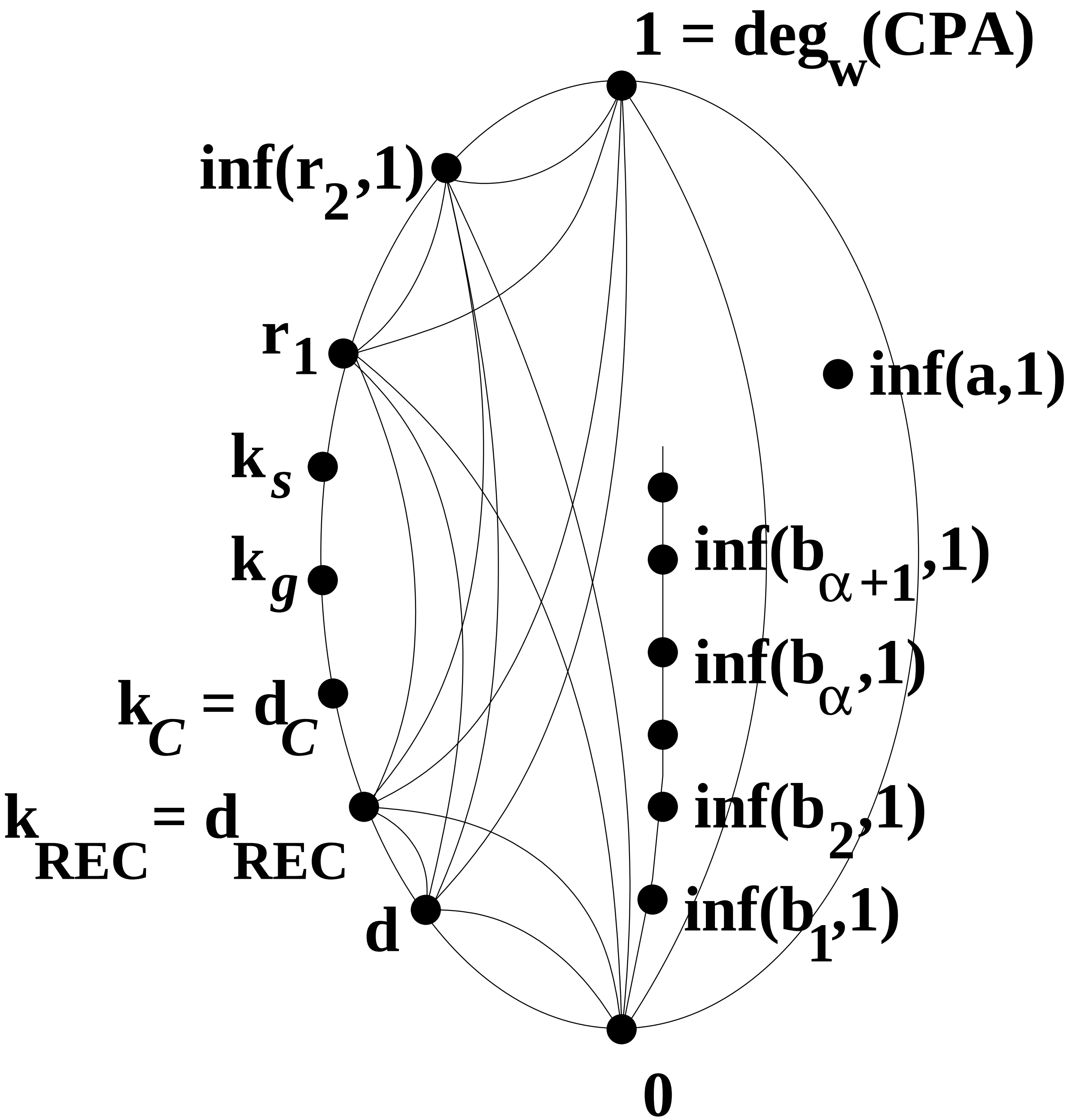
For $x \in 2^{\mathbb{N}}$ let $\mathrm{effdim}(x) =$ the *effective Hausdorff dimension* of x, i.e.,

$$\mathrm{effdim}(x) = \liminf_{n \to \infty} \frac{\mathsf{K}(x{\restriction}\{1, \ldots, n\})}{n}.$$

Given a right recursively enumerable real number $s < 1$, we have $\mathbf{k}_s \in \mathcal{E}_{\mathsf{w}}$ where

$$\mathbf{k}_s = \deg(\{x \in 2^{\mathbb{N}} \mid \mathrm{effdim}(x) > s\}).$$

Moreover, $s < t$ implies $\mathbf{k}_s < \mathbf{k}_t$ (Miller).



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More generally, let $g : \mathbb{N} \to [-\infty, \infty)$ be an unbounded computable function such that $g(n) \leq g(n+1) \leq g(n) + 1$ for all n.

For example, $g(n)$ could be $n/2$ or $n/3$ or $\sqrt{n}$ or $\sqrt[3]{n}$ or $\log n$ or $\log n + \log \log n$ or $\log \log n$ or the inverse Ackermann function.

Define $\mathbf{k}_g = \deg(\{x \in 2^{\mathbb{N}} \mid x$ is *g-complex*$\})$, i.e., $\exists c \forall n\, (\mathsf{K}(x{\upharpoonright}\{1, \ldots, n\} \geq g(n) - c)$.

Theorem (Hudelson 2010)**.** We have $\mathbf{k}_g < \mathbf{k}_h$ provided $g(n) + 2 \log g(n) \leq h(n)$ for all n.

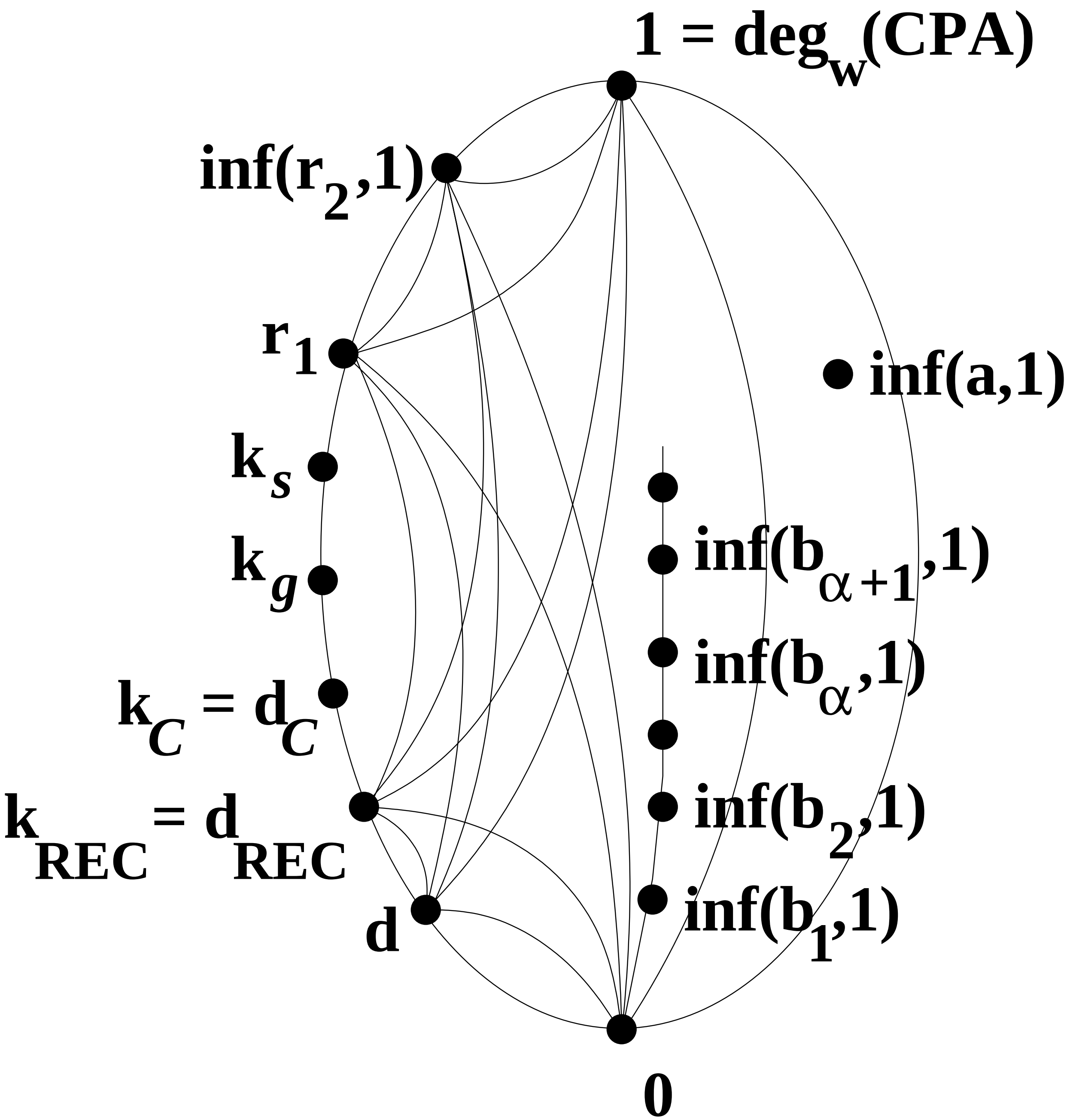
In other words, there exists a g-complex real with no h-complex real Turing reducible to it.

This is a generalization of Miller's theorem on the difficulty of information extraction.

References:

Phil Hudelson, Mass problems and initial segment complexity, in preparation.

Joseph S. Miller, Extracting information is hard, Advances in Mathematics, 226, 2011, 373–384.



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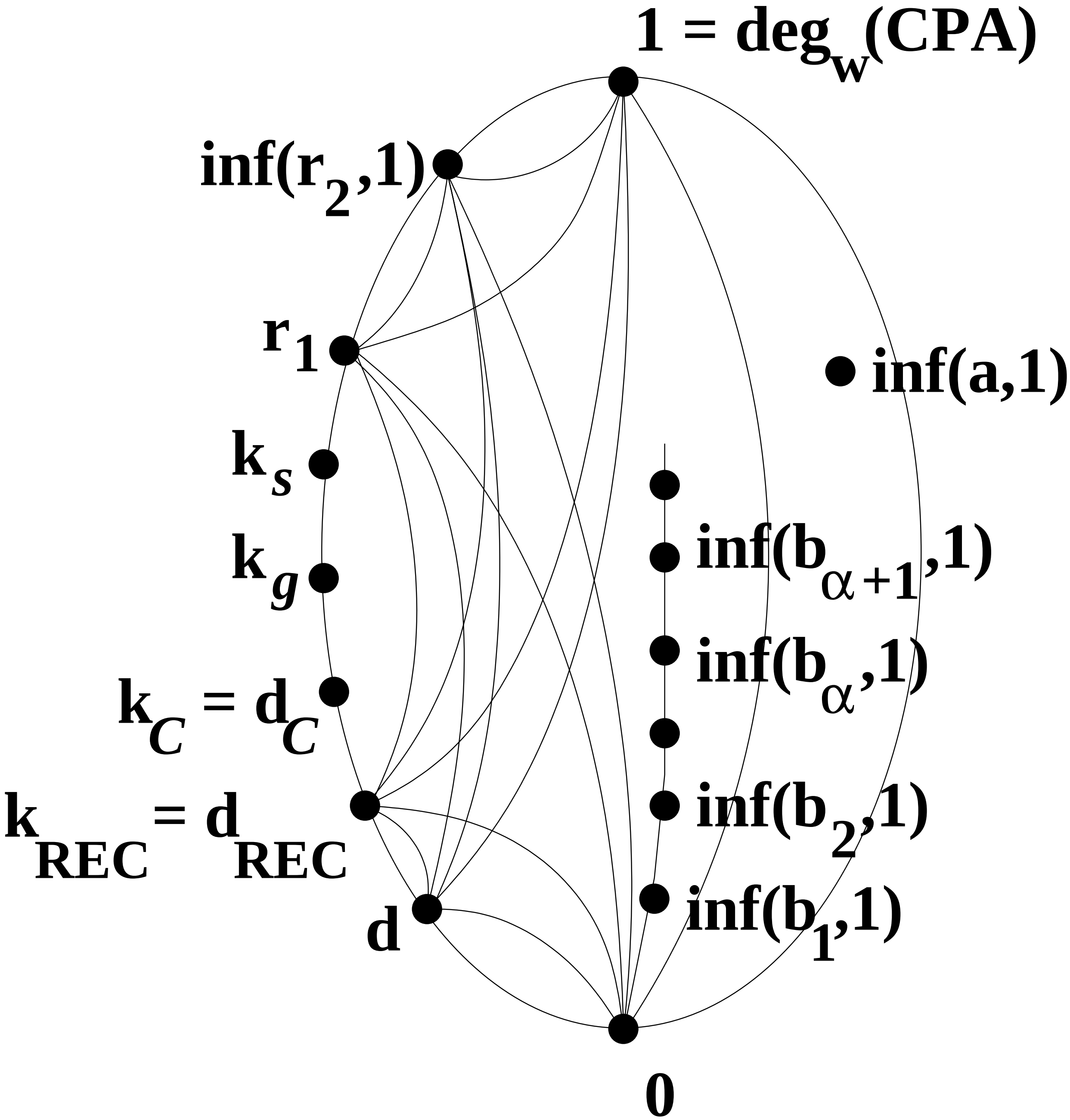
Letting z be a Turing oracle, define
$\mathsf{MLR}^z = \{x \in 2^{\mathbb{N}} \mid x$ is random relative to $z\}$
and $\mathsf{K}^z(\tau) =$ the prefix-free Kolmogorov complexity of τ relative to z.

Define $y \leq_{\mathsf{LR}} z \iff \mathsf{MLR}^z \subseteq \mathsf{MLR}^y$
and $y \leq_{\mathsf{LK}} z \iff \exists c \, \forall \tau \, (\mathsf{K}^z(\tau) \leq \mathsf{K}^y(\tau) + c)$.

Theorem (Miller/Kjos-Hanssen/Solomon)**.**
We have $y \leq_{\mathsf{LR}} z \iff y \leq_{\mathsf{LK}} z$.

For each recursive ordinal number α, let $0^{(\alpha)} =$ the αth iterated Turing jump of 0. Thus $0^{(1)} =$ the halting problem, and $0^{(\alpha+1)} =$ the halting problem relative to $0^{(\alpha)}$, etc. This is the hyperarithmetical hierarchy. We embed it naturally into $\mathcal{E}_{\mathsf{W}}$ as follows.

Theorem (Simpson 2009)**.** $0^{(\alpha)} \leq_{\mathsf{LR}} z$
$\iff$ every $\Sigma^0_{\alpha+2}$ set includes a $\Sigma^{0,z}_2$ set of the same measure. Moreover, letting $\mathbf{b}_\alpha = \deg(\{z \mid 0^{(\alpha)} \leq_{\mathsf{LR}} z\})$ we have
$\inf(\mathbf{b}_\alpha, \mathbf{1}) \in \mathcal{E}_{\mathsf{W}}$ and $\inf(\mathbf{b}_\alpha, \mathbf{1}) < \inf(\mathbf{b}_{\alpha+1}, \mathbf{1})$.



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History: Kolmogorov 1932 developed his "calculus of problems" as a nonrigorous yet compelling explanation of Brouwer's intuitionism. Later Medvedev 1955 and Muchnik 1963 proposed Medvedev degrees and Muchnik degrees as rigorous versions of Kolmogorov's idea.

Some references:

Stephen G. Simpson, Mass problems and randomness, Bulletin of Symbolic Logic, 11, 2005, 1–27.

Stephen G. Simpson, An extension of the recursively enumerable Turing degrees, Journal of the London Mathematical Society, 75, 2007, 287–297.

Stephen G. Simpson, Mass problems and intuitionism, Notre Dame Journal of Formal Logic, 49, 2008, 127–136.

Stephen G. Simpson, Mass problems and measure-theoretic regularity, Bulletin of Symbolic Logic, 15, 2009, 385–409.

Stephen G. Simpson, Medvedev degrees of 2-dimensional subshifts of finite type, to appear in Ergodic Theory and Dynamical Systems.

Stephen G. Simpson, Entropy equals dimension equals complexity, 2010, in preparation.

THE END. THANK YOU!