Mass Problems

Stephen G. Simpson

Pennsylvania State University
NSF grant DMS-0600823
http://www.math.psu.edu/simpson/

simpson@math.psu.edu

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Motivation:

Recall that $\mathcal{D}_T$ is the upper semilattice of all Turing degrees.

In $\mathcal{D}_T$ there are a great many specific, interesting Turing degrees, namely

$$\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \cdots < \mathbf{0}^{(\alpha)} < \mathbf{0}^{(\alpha+1)} < \cdots$$

where α runs through (a large initial segment of) the countable ordinal numbers (depending on whether V=L or not . . .). See my paper *The hierarchy based on the jump operator*, Kleene Symposium, North-Holland, 1980.

Historically, the *original purpose* of $\mathcal{D}_T$ (Turing 1936, Kleene/Post 1940's, 1950's) was to serve as a framework for classifying unsolvable mathematical problems. Recall the famous phrase

"degrees of unsolvability".

In the 1950's, 1960's, and 1970's, it turned out that many specific, natural, well-known, unsolvable mathematical problems are indeed of Turing degree $\mathbf{0}'$:

- the Halting Problem for Turing machines (Turing's original example)

- the Word Problem for finitely presented groups

- the Triviality Problem for finitely presented groups, etc.

- Hilbert's 10th Problem for Diophantine equations

- and many others.

In addition, the **arithmetical hierarchy**

$$\mathbf{0}^{(n)}, \quad n < \omega$$

and the **hyperarithmetical hierarchy**

$$\mathbf{0}^{(\alpha)}, \quad \alpha < \omega_1^{\mathsf{CK}}$$

have been useful in studying the foundations of mathematics.

These hierarchies, based on iterating the Turing jump operator, have been useful precisely because of their ability to classify unsolvable mathematical problems.

This aspect of $\mathcal{D}_T$ is explored in my book *Subsystems of Second Order Arithmetic*, Springer-Verlag, 1999, which is the basic reference on reverse mathematics.

On the other hand, there are many parts of $\mathcal{D}_T$ which seem largely irrelevant to foundations of mathematics.

For example, there is $\mathcal{E}_T$, the upper semilattice of recursively enumerable Turing degrees. Many structural and methodological aspects of $\mathcal{E}_T$ have received a huge amount of attention over many decades. However, the vast resources spent on $\mathcal{E}_T$ have yielded nothing when it comes to foundations of mathematics and the classification of unsolvable mathematical problems.

Moreover, there are many unsolvable mathematical problems which do not fit into the $\mathcal{D}_T$ framework at all.

For example, . . .

Many unsolvable mathematical problems do not fit into the Turing degree framework.

For example, consider the following problem, which we call PA:

> To find a complete, consistent theory which includes Peano Arithmetic.

Note that PA is a very natural problem, in view of the Gödel Incompleteness Theorem, which says that Peano Arithmetic itself is incomplete.

Moreover, by the work of Tarski, Gödel, and Rosser, the problem PA is "unsolvable" in the sense that there is is no *computable*, complete, consistent theory which includes Peano Arithmetic.

The scandal here is that it is not possible to assign a specific Turing degree ("degree of unsolvability") to the unsolvable problem PA.

PA is this unsolvable problem:

> To find a complete, consistent theory which includes Peano Arithmetic.

Although PA is unsolvable, *there is no one specific Turing degree* associated to PA. Thus, the Turing degree framework fails to classify PA.

Digression: One may consider the Turing degree $\mathbf{0}^{(\omega)}$. It is reasonable to associate $\mathbf{0}^{(\omega)}$ to True Arithmetic, which is one particular, complete, consistent extension of Peano Arithmetic. However, it is unreasonable to associate $\mathbf{0}^{(\omega)}$ to the problem PA as a whole. This is because, beyond True Arithmetic, there are many other complete, consistent extensions of Peano Arithmetic. Some of them even have Turing degree $< \mathbf{0}'$.

If we want to classify unsolvable problems such as PA, we need a different framework.

The appropriate framework is:

MASS PROBLEMS.

Here are some more examples.

R_1: To find an infinite sequence of 0's and 1's which is *random* in the sense of Martin-Löf.

R_n: To find an infinite sequence of 0's and 1's which is n-*random*, $n = 1, 2, \ldots$.

DNR: To find a function f which is *diagonally nonrecursive*, i.e., $f(n) \neq \varphi_n^{(1)}(n)$ for all n.

$\mathsf{DNR}_{\mathsf{REC}}$: To find a function f which is diagonally nonrecursive and *recursively dominated*, i.e., there exists a recursive function g such that $f(n) < g(n)$ for all but finitely many n.

AED: To find a Turing oracle A which is *almost everwhere dominating*, i.e., with probability 1, every function which is computable from a sequence of coin tosses is dominated by some function which is computable from A.

Each of the problems R_1, R_2, . . . , DNR, $\mathsf{DNR}_{\mathsf{REC}}$, AED, . . . , is similar to the problem PA. In each case, the problem is *unsolvable* (i.e., there is no computable solution), but *there is no one specific Turing degree* that can be attached to the problem.

In other words, each of the problems

$$\text{PA}, \text{R}_1, \text{R}_2, \ldots, \text{DNR}, \text{DNR}_{\text{REC}}, \text{AED}, \ldots,$$

is an example of an *unsolvable mass problem*.

If we wish to classify unsolvable problems of this kind, we need a concept of "degree of unsolvability" which is more general than the Turing degrees.

The appropriately generalized concept of "degree of unsolvability" is:

WEAK DEGREES,

also known as

MUCHNIK DEGREES.

This concept of "degree of unsolvability" is the one that has turned out to be most useful for classification and comparison of specific, natural, unsolvable mass problems.

More motivation:

Recall that $\mathcal{E}_T$ is the upper semilattice of recursively enumerable Turing degrees.

Two basic, classical, unresolved issues concerning $\mathcal{E}_T$ are:

Issue 1: To find a specific, natural, r.e. Turing degree $\mathbf{a} \in \mathcal{E}_T$ which is $> \mathbf{0}$ and $< \mathbf{0}'$.

Issue 2: To find a "smallness property" of an infinite co-r.e. set $A \subseteq \omega$ which insures that $\deg_T(A) = \mathbf{a} \in \mathcal{E}_T$ is $> \mathbf{0}$ and $< \mathbf{0}'$.

These unresolved issues go back to Post's 1944 paper, *Recursively enumerable sets of positive integers and their decision problems*.

Mass Problems to the Rescue!

We address Issues 1 and 2 by passing from *decision problems* to *mass problems*.

Outline of this talk:

We embed $\mathcal{E}_T$ into a slightly larger structure, $\mathcal{P}_w$, which is much better behaved. In the $\mathcal{P}_w$ context, we obtain satisfactory, positive answers to Issues 1 and 2.

What is this wonderful structure $\mathcal{P}_w$?

Briefly, $\mathcal{P}_w$ is the lattice of weak degrees of mass problems associated with nonempty Π^0_1 subsets of 2^ω.

In order to explain $\mathcal{P}_w$, we must first explain:

- mass problems,
- weak degrees, and
- nonempty Π^0_1 subsets of 2^ω.

Mass problems (informal discussion):

A "decision problem" is the problem of deciding whether a given $n \in \omega$ belongs to a fixed set $A \subseteq \omega$ or not. To compare decision problems, we use Turing reducibility. $A \leq_T B$ means that A can be computed using an oracle for B.

A "mass problem" is a problem with a not necessarily unique solution. (By contrast, a "decision problem" has only one solution.)

The "mass problem" associated with a set $P \subseteq \omega^\omega$ is the "problem" of computing an element of P.

The "solutions" of P are the elements of P.

One mass problem is said to be "reducible" to another if, given any solution of the second problem, we can use it as an oracle to compute a solution of the first problem.

Rigorous definition:

Let P and Q be subsets of ω^ω.

We view P and Q as mass problems.

We say that P is *weakly reducible* to Q if

$$(\forall Y \in Q)\ (\exists X \in P)\ (X \leq_T Y)\ .$$

This is abbreviated $P \leq_w Q$.

Summary:

$P \leq_w Q$ means that, given any solution of Q, we can use it as an oracle to compute a solution of P.

Digression: weak vs. strong reducibility

Let P and Q be subsets of ω^ω.

1. P is *weakly reducible* to Q, $P \leq_w Q$, if for all $Y \in Q$ there exists e such that $\{e\}^Y \in P$.

2. P is *strongly reducible* to Q, $P \leq_s Q$, if there exists e such that $\{e\}^Y \in P$ for all $Y \in Q$.

Strong reducibility is a uniform variant of weak reducibility. By a result of Nerode, there is an analogy:

$$\frac{\text{weak reducibility}}{\text{Turing reducibility}} = \frac{\text{strong reducibility}}{\text{truth table reducibility}}.$$

In this talk we deal only with weak reducibility.

Historical note:

Weak reducibility is due to Muchnik 1963.

Strong reducibility is due to Medvedev 1955.

The lattice $\mathcal{P}_w$:

We focus on Π^0_1 subsets of 2^ω, i.e., $P = \{\text{paths through } T\}$ where T is a recursive subtree of $2^{<\omega}$, the full binary tree of finite sequences of 0's and 1's.

We define $\mathcal{P}_w$ to be the set of weak degrees of nonempty Π^0_1 subsets of 2^ω, ordered by weak reducibility.

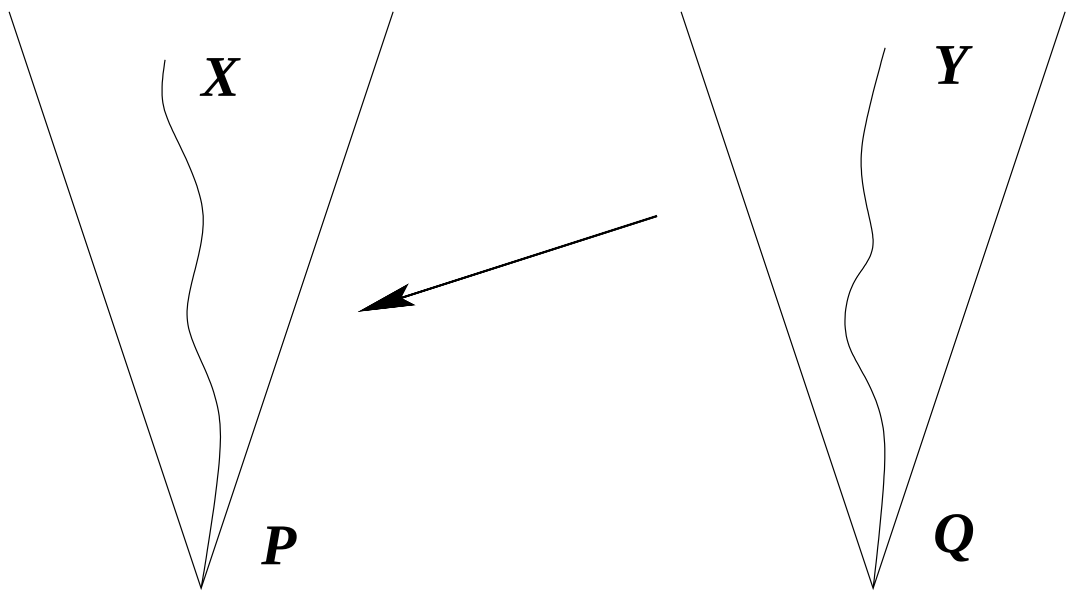
Basic facts about $\mathcal{P}_w$:

1. $\mathcal{P}_w$ is a distributive lattice, with l.u.b. given by $P \times Q = \{X \oplus Y \mid X \in P, Y \in Q\}$, and g.l.b. given by $P \cup Q$.

2. The bottom element of $\mathcal{P}_w$ is the weak degree of 2^ω.

3. The top element of $\mathcal{P}_w$ is the weak degree of PA = {completions of Peano Arithmetic}. (Scott/Tennenbaum).

Weak reducibility of Π^0_1 subsets of 2^ω:



$P \leq_w Q$ means:
$(\forall Y \in Q)\ (\exists X \in P)\ (X \leq_T Y)$.

P, Q are given by infinite recursive subtrees of the full binary tree of finite sequences of 0's and 1's.

X, Y are infinite (nonrecursive) paths through P, Q respectively.

The lattice $\mathcal{P}_w$ (review):

A *weak degree* is an equivalence class of subsets of ω^ω under the equivalence relation $P \leq_w Q$ and $Q \leq_w P$. The weak degrees have a partial ordering induced by $\leq_w$.

We define $\mathcal{P}_w$ to be the set of weak degrees of nonempty Π^0_1 subsets of 2^ω, partially ordered by weak reducibility.

$\mathcal{P}_w$ is a countable distributive lattice.

The bottom element of $\mathcal{P}_w$ is the weak degree of 2^ω.

The top element of $\mathcal{P}_w$ is the weak degree of

$$\mathrm{PA} = \{\text{completions of Peano Arithmetic}\}.$$

Embedding $\mathcal{E}_T$ into $\mathcal{P}_w$:

Theorem (Simpson 2002):

There is a natural embedding $\phi : \mathcal{E}_T \to \mathcal{P}_w$.

($\mathcal{E}_T =$ the semilattice of Turing degrees of r.e. subsets of ω. $\mathcal{P}_w =$ the lattice of weak degrees of nonempty Π^0_1 subsets of 2^ω.)

The embedding ϕ is given by

$$\phi : \text{deg}_T(A) \mapsto \text{deg}_w(\text{PA} \cup \{A\}).$$

Note that $\text{PA} \cup \{A\}$ is not a Π^0_1 set. However, it is of the same weak degree as a Π^0_1 set. This is already a nontrivial result.

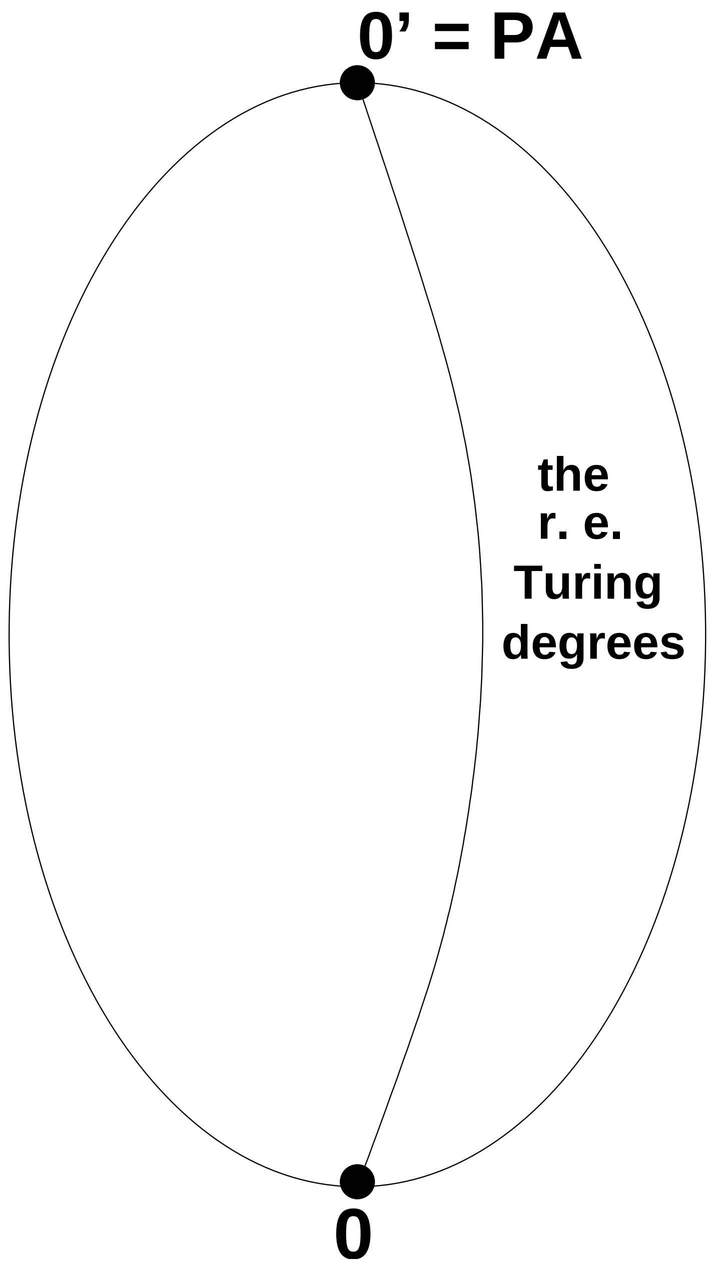
The embedding ϕ is one-to-one and preserves $\leq$, l.u.b., and the top and bottom elements.

Convention:

We identify $\mathcal{E}_T$ with its image in $\mathcal{P}_w$ under ϕ.

In particular, we identify $\mathbf{0}', \mathbf{0} \in \mathcal{E}_T$ with the top and bottom elements of $\mathcal{P}_w$.

A picture of the lattice $\mathcal{P}_w$:



$\mathcal{E}_T$ is embedded in $\mathcal{P}_w$. $\mathbf{0}'$ and $\mathbf{0}$ are the top and bottom elements of both $\mathcal{E}_T$ and $\mathcal{P}_w$.

Structural properties of $\mathcal{P}_w$:

1. $\mathcal{P}_w$ is a countable distributive lattice. Every countable distributive lattice is lattice embeddable in every initial segment of $\mathcal{P}_w$. (Binns/Simpson 2001)

2. The $\mathcal{P}_w$ analog of the Sacks Splittting Theorem holds. (Binns, 2002)

3. We conjecture that the $\mathcal{P}_w$ analog of the Sacks Density Theorem holds.

These structural results for $\mathcal{P}_w$ are proved by means of priority arguments, just as for $\mathcal{E}_T$.

4. Within $\mathcal{P}_w$ the degrees $\mathbf{r}_1$ and $\inf(\mathbf{r}_2, \mathbf{0}')$ are meet irreducible and do not join to $\mathbf{0}'$. (Simpson 2002, 2004)

5. $\mathbf{0}$ is meet irreducible. (This is trivial.)

Response to Issue 1:

Issue 1 was:

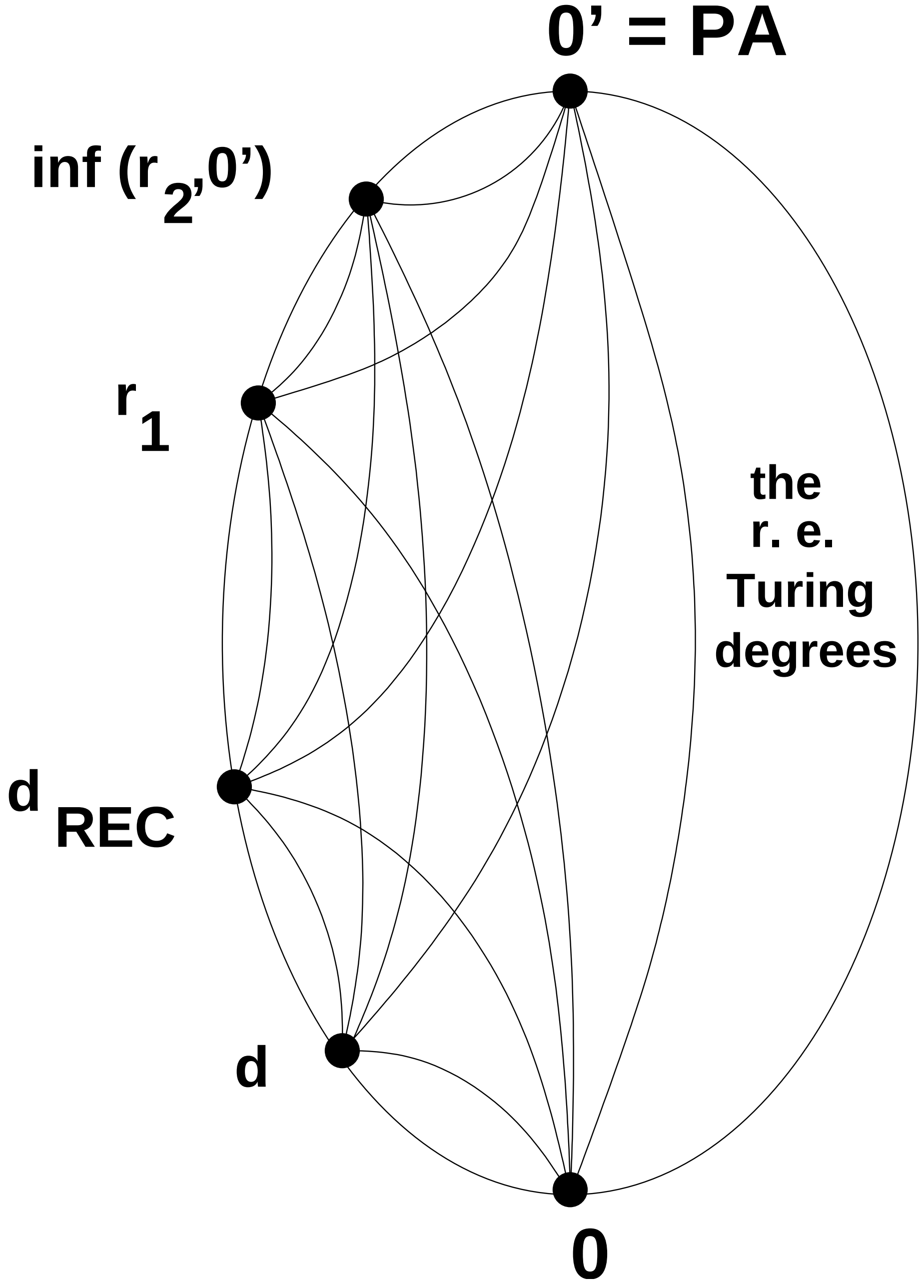
To find a specific, natural example of a recursively enumerable Turing degree which is $> \mathbf{0}$ and $< \mathbf{0}'$.

We do not know how to do this.

However, in the $\mathcal{P}_w$ context, we have discovered many specific, natural degrees which are $> \mathbf{0}$ and $< \mathbf{0}'$.

The specific, natural degrees in $\mathcal{P}_w$ which we have discovered are related to foundationally interesting topics:

- algorithmic randomness,
- diagonal nonrecursiveness,
- reverse mathematics,
- subrecursive hierarchies,
- computational complexity.



Note: Except for $\mathbf{0}'$ and $\mathbf{0}$, the r.e. Turing degrees are incomparable with these specific, natural degrees in $\mathcal{P}_w$.

Some specific, natural degrees in $\mathcal{P}_w$:

$\mathbf{r}_n =$ the weak degree of the set of n-random reals.

$\mathbf{d} =$ the weak degree of the set of diagonally nonrecursive functions.

$\mathbf{d}_{\mathsf{REC}} =$ the weak degree of the set of diagonally nonrecursive functions which are recursively bounded.

Theorem (Simpson 2002, Ambos $\cdots$ 2004)**:**
In $\mathcal{P}_w$ we have

$$\mathbf{0} < \mathbf{d} < \mathbf{d}_{\mathsf{REC}} < \mathbf{r}_1 < \inf(\mathbf{r}_2, \mathbf{0}') < \mathbf{0}'.$$

Theorem (Simpson 2004)**:**

1. $\mathbf{r}_1$ is the maximum weak degree of a Π^0_1 subset of 2^ω which is of positive measure.

2. $\inf(\mathbf{r}_2, \mathbf{0}')$ is the maximum weak degree of a Π^0_1 subset of 2^ω whose Turing upward closure is of positive measure.

Structural properties of $\mathcal{P}_w$:

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2. The $\mathcal{P}_w$ analog of the Sacks Splittting Theorem holds. (Stephen Binns, 2002)

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These structural results for $\mathcal{P}_w$ are proved by means of priority arguments, just as for $\mathcal{E}_T$.

4. Within $\mathcal{P}_w$ the degrees $\mathbf{r}_1$ and $\inf(\mathbf{r}_2, \mathbf{0}')$ are meet irreducible and do not join to $\mathbf{0}'$. (Simpson 2002, 2004)

5. $\mathbf{0}$ is meet irreducible. (This is trivial.)

Another source of specific degrees in $\mathcal{P}_w$: almost everywhere domination.

Definition (Dobrinen/Simpson 2004)**:**

B is *almost everywhere dominating* if, for almost all $X \in 2^\omega$, each function $\leq_T X$ is dominated by some function $\leq_T B$.

Here "almost all" refers to the fair coin measure on 2^ω.

Randomness and a.e. domination are closely related to the reverse mathematics of measure theory.

Some additional, natural degrees in $\mathcal{P}_w$:

Let $\mathbf{b}_1 = \text{deg}_w(\text{AED})$ where

$$\text{AED} = \{B \mid B \text{ is a. e. dominating}\}.$$

Let $\mathbf{b}_2 = \text{deg}_w(\text{AED} \times \text{R}_1)$ where

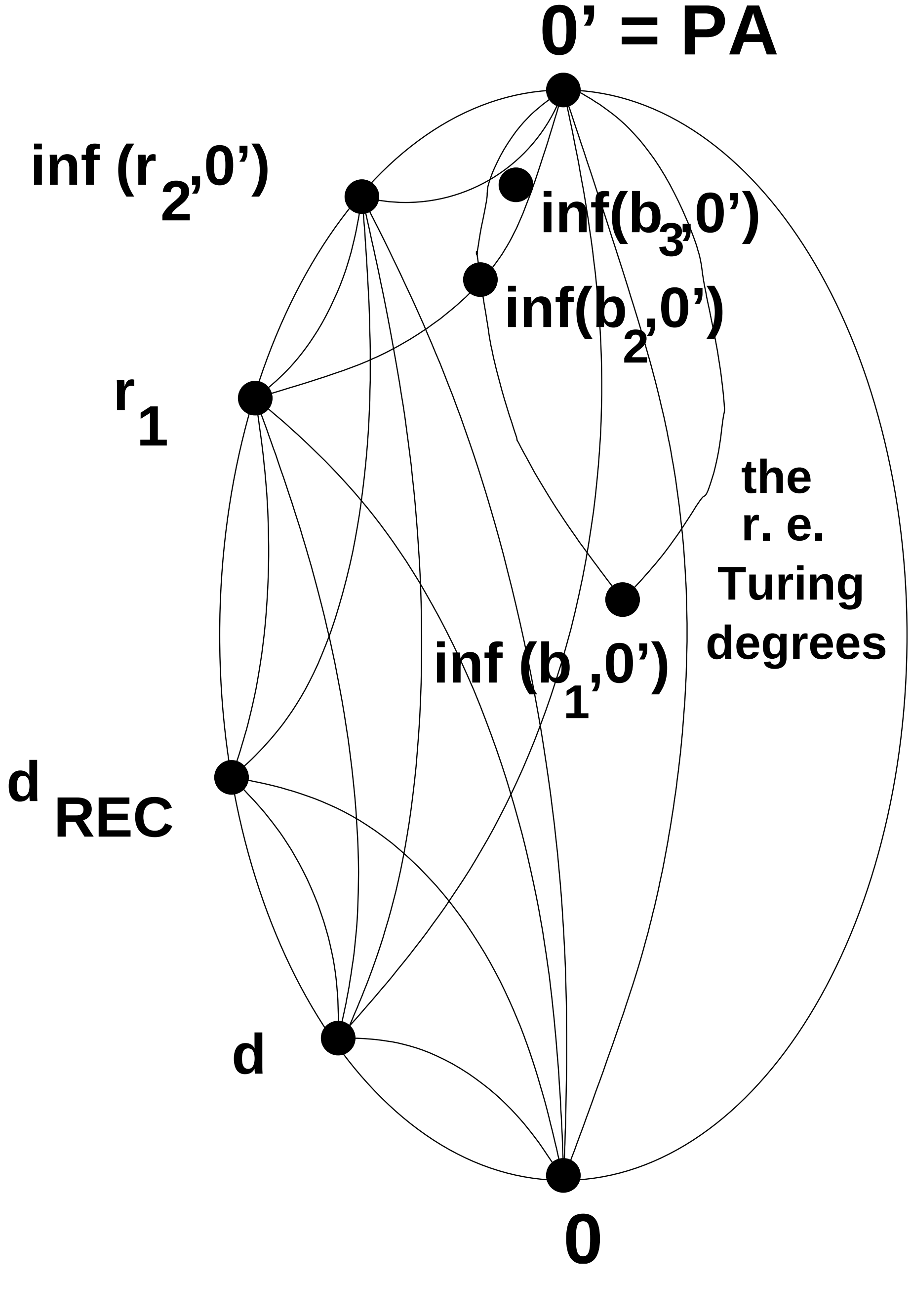
$$\text{R}_1 = \{A \mid A \text{ is 1-random}\}.$$

Let $\mathbf{b}_3 = \text{deg}_w(\text{AED} \cap \text{R}_1)$.

Theorem (Simpson, 2006)**:** In $\mathcal{P}_w$ we have:

- $\mathbf{0} < \inf(\mathbf{b}_1, \mathbf{0}') < \inf(\mathbf{b}_2, \mathbf{0}') < \inf(\mathbf{b}_3, \mathbf{0}') < \mathbf{0}'$.
- $\inf(\mathbf{b}_1, \mathbf{0}') <$ some r.e. degrees $< \mathbf{0}'$.
- $\inf(\mathbf{b}_2, \mathbf{0}') \mid$ all r.e. degrees except $\mathbf{0}, \mathbf{0}'$.
- $\inf(\mathbf{b}_3, \mathbf{0}') >$ some r.e. degrees $> \mathbf{0}$.

The proof uses virtually everything that is known about randomness and almost everywhere domination (Cholak, Greenberg, Miller, Binns, Kjos-Hanssen, Lerman, Solomon, Hirschfeldt, Nies, . . .).



Note that inf($\mathbf{b}_1, \mathbf{0}'$) and inf($\mathbf{b}_3, \mathbf{0}'$), unlike inf($\mathbf{b}_2, \mathbf{0}'$), are comparable with some r.e. Turing degrees other than $\mathbf{0}'$ and $\mathbf{0}$.

Some additional, specific degrees in $\mathcal{P}_w$:

Definition: $\mathbf{d}_{\mathsf{REC}}$ = the weak degree of the set of recursively bounded DNR functions.

Theorem (Kjos-Hanssen/Merkle/Stephan): $\mathbf{d}_{\mathsf{REC}}$ = the weak degree of
$\{A \in 2^\omega \mid (\exists f \in \mathsf{REC})\, \forall n\, (K(A \upharpoonright n) > f^{-1}(n))\}$.
Here K denotes Kolmogorov complexity.

Definition (Simpson 2004): $\mathbf{d}_\alpha$ = the weak degree of the set of DNR functions bounded by some $f \in \mathsf{REC}_\alpha$. Here REC_α is the Wainer hierarchy, $\alpha \leq \varepsilon_0$.

Theorem (Kjos-Hanssen/Simpson 2006): $\mathbf{d}_\alpha$ = the weak degree of
$\{A \mid (\exists f \in \mathsf{REC}_\alpha)\, \forall n\, (K(A \upharpoonright n) > f^{-1}(n))\}$.

Ambos-Spies/Kjos-Hanssen/Lempp/Slaman 2004 and Simpson 2005 have shown that in $\mathcal{P}_w$ we have

$$\mathbf{r}_1 > \mathbf{d}_0 > \mathbf{d}_1 > \cdots > \mathbf{d}_\alpha > \cdots > \mathbf{d}_{\mathsf{REC}}.$$

Some additional examples in $\mathcal{P}_w$:

Definition:
$\mathbf{d}^2 =$ the weak degree of the set of $f \oplus g$ such that f is diagonally nonrecursive, and g is diagonally nonrecursive relative to f. More generally, define $\mathbf{d}^n$ for all $n \geq 1$. This can be extended into the transfinite.

Theorem (A-S/K-H/L/S, Simpson)**:**
In $\mathcal{P}_w$ we have

$$\mathbf{r}_1 > \mathbf{d}_0 > \mathbf{d}_1 > \cdots > \mathbf{d}_\alpha > \cdots > \mathbf{d}_{\mathsf{REC}}$$

and

$$\mathbf{d} = \mathbf{d}^1 < \mathbf{d}^2 < \cdots < \mathbf{d}^n < \cdots < \mathbf{r}_1 \,.$$

We conjecture that $\mathbf{d}^n$ is incomparable with $\mathbf{d}_\alpha$ and with $\mathbf{d}_{\mathsf{REC}}$. This would be another example of specific, natural degrees in $\mathcal{P}_w$ which are incomparable with each other.

Index sets in $\mathcal{P}_w$:

Here is a result indicating that $\mathcal{P}_w$ partakes of hyperarithmeticity.

Let P_i, $i \in \omega$ be the standard enumeration of all nonempty Π^0_1 subsets of 2^ω.
Let $\mathbf{p}_i = \deg_w(P_i)$.
By definition, $\mathcal{P}_w = \{\mathbf{p}_i \mid i \in \omega\}$.

Theorem (Cole/Simpson 2006)**:**
The index set $\{i \mid \mathbf{p}_i = \mathbf{0}'\}$ is $\Pi^1_1 complete$.

More generally:

Theorem (Cole/Simpson 2006)**:**
For any j such that $\mathbf{p}_j > \mathbf{0}$, the index sets $\{i \mid \mathbf{p}_i = \mathbf{p}_j\}$ and $\{i \mid \mathbf{p}_i \geq \mathbf{p}_j\}$ are Π^1_1 complete.

Problem: Characterize the j's for which $\{i \mid \mathbf{p}_i \leq \mathbf{p}_j\}$ is Π^1_1 complete.

Embedding hyperarithmeticity into $\mathcal{P}_w$:

Definition (Cole/Simpson 2006):
A function $f(n)$ is said to be *boundedly limit recursive* in a Turing oracle A, abbreviated $f \in \mathsf{BLR}(A)$, if there exist an A-recursive approximating function $\widetilde{f}(n,s)$ and a recursive bounding function $\widehat{f}(n)$ such that for all n, $f(n) = \lim_s \widetilde{f}(n,s)$ and $|\{s \mid \widetilde{f}(n,s) \neq \widetilde{f}(n,s+1)\}| < \widehat{f}(n)$.

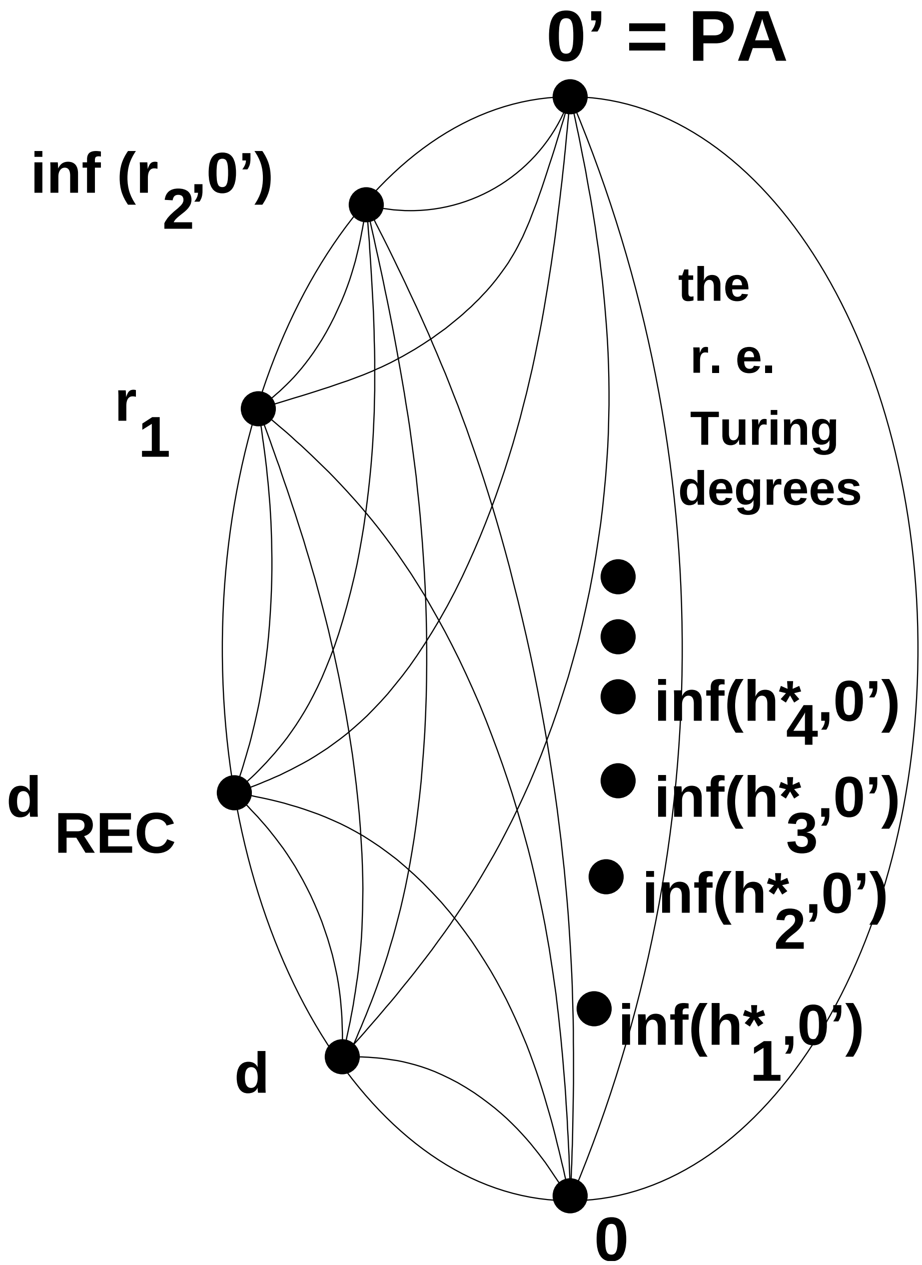
Definition (Cole/Simpson 2006):
For $\alpha < \omega_1^{\mathsf{CK}}$ let $\mathbf{h}_\alpha^* =$ the weak degree of

$$\{A \mid \mathsf{BLR}(0^{(\alpha)}) \subseteq \mathsf{BLR}(A)\}.$$

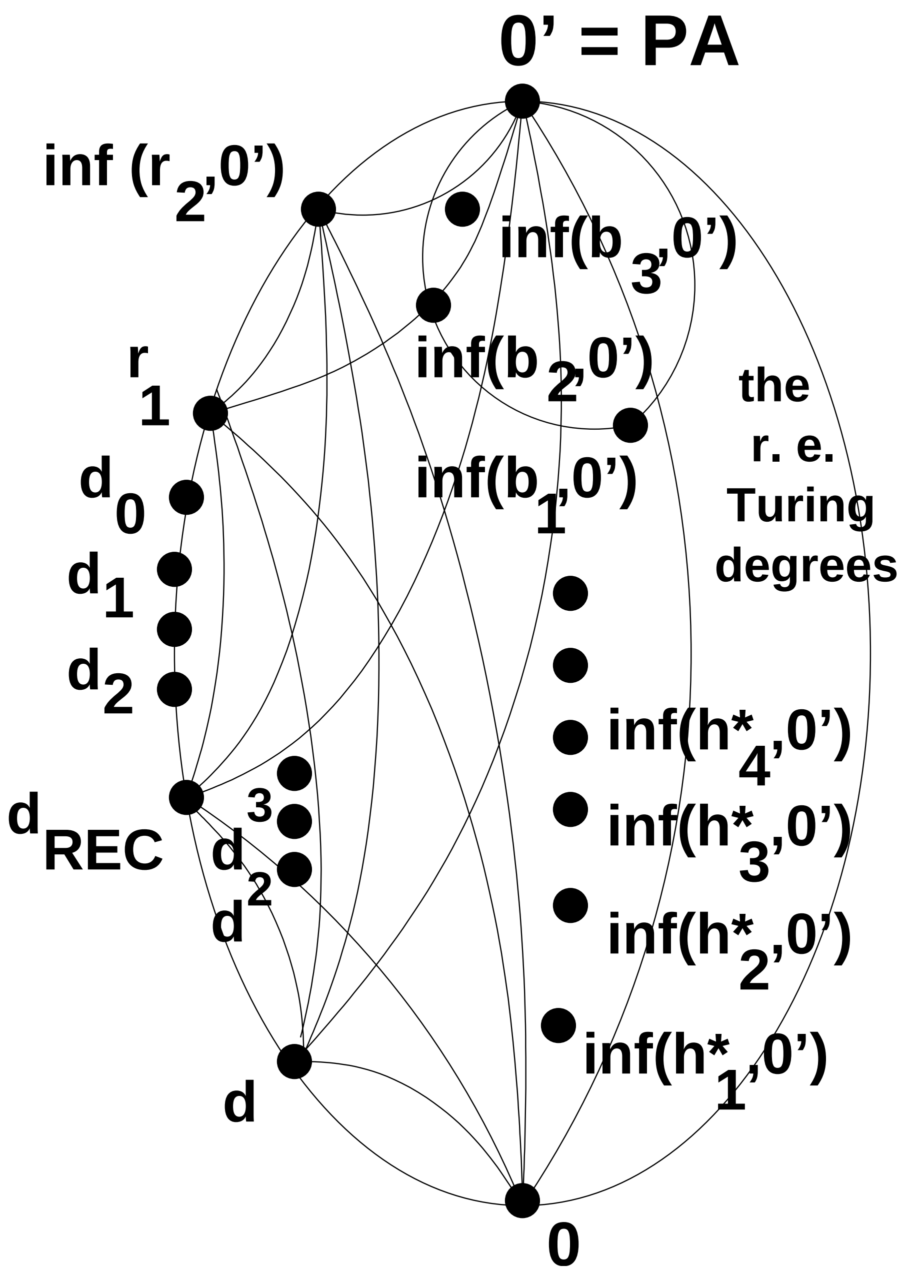
Theorem (Cole/Simpson 2006):
In $\mathcal{P}_w$ we have

$$\mathbf{0} < \inf(\mathbf{h}_1^*, \mathbf{0}') < \inf(\mathbf{h}_2^*, \mathbf{0}') < \cdots$$
$$< \inf(\mathbf{h}_\alpha^*, \mathbf{0}') < \inf(\mathbf{h}_{\alpha+1}^*, \mathbf{0}') < \cdots < \mathbf{0}'$$

and these weak degrees are incomparable with $\mathbf{d}$, $\mathbf{d}_{\mathsf{REC}}$, $\mathbf{r}_1$, $\inf(\mathbf{r}_2, \mathbf{0}')$ and all recursively enumerable Turing degrees except $\mathbf{0}$ and $\mathbf{0}'$.



The weak degrees $\inf(\mathbf{h}^*_\alpha, \mathbf{0}')$, $1 \le \alpha < \omega_1^{\mathsf{CK}}$, are incomparable with $\mathbf{d}$, $\mathbf{d}_{\mathsf{REC}}$, $\mathbf{r}_1$, $\inf(\mathbf{r}_2, \mathbf{0}')$, and all r.e. Turing degrees except $\mathbf{0}$ and $\mathbf{0}'$.



A more comprehensive picture of $\mathcal{P}_w$.
$\mathbf{r} =$ randomness, $\mathbf{h} =$ hyperarithmeticity,
$\mathbf{b} =$ almost everywhere domination,
$\mathbf{d} =$ diagonal nonrecursiveness, etc.

Definition. $S \subseteq \omega^\omega$ is Σ^0_3 if

$$S = \{f \in \omega^\omega \mid \exists i \, \forall m \, \exists n \, R(i,m,n,f)\}$$

for some recursive predicate $R \subseteq \omega^3 \times \omega^\omega$.

Many interesting mass problems are Σ^0_3.

Examples:

- R_1 is Σ^0_2.
- R_2 is Σ^0_3.
- DNR is Π^0_1.
- $\mathsf{DNR}_{\mathsf{REC}}$ is Σ^0_3.
- AED is Σ^0_3.

The Embedding Lemma:

If $S \subseteq \omega^\omega$ *is* Σ^0_3 *and if* $P \subseteq 2^\omega$ *is nonempty* Π^0_1*, then* $\deg_w(S \cup P) \in \mathcal{P}_w$.

It follows that, for many Σ^0_3 sets $S \subseteq \omega^\omega$, $\deg_w(S) \in \mathcal{P}_w$.

Examples:

1. $\mathsf{R}_1 = \{X \in 2^\omega \mid X \text{ is 1-random}\}$.

Since R_1 is Σ^0_2, it follows by the Embedding Lemma that $\mathbf{r}_1 = \deg_w(R_1) \in \mathcal{P}_w$.

2. $\mathsf{R}_2 = \{X \in 2^\omega \mid X \text{ is 2-random}\}$.

Since R_2 is Σ^0_3, it follows by the Embedding Lemma that $\inf(\mathbf{r}_2, \mathbf{0}') = \deg_w(R_2 \cup \mathsf{PA}) \in \mathcal{P}_w$.

3. $\mathsf{D} = \{f \in \omega^\omega \mid f \text{ is diagonally nonrecursive}\}$.

Since D is Π^0_1, $\mathbf{d} = \deg_w(D) \in \mathcal{P}_w$.

4. $\mathsf{D}_{\mathsf{REC}} = \{f \in D \mid f \text{ is recursively bounded}\}$.

Since $\mathsf{D}_{\mathsf{REC}}$ is Σ^0_3, $\mathbf{d}_{\mathsf{REC}} = \deg_w(\mathsf{D}_{\mathsf{REC}}) \in \mathcal{P}_w$.

5. Let $A \subseteq \omega$ be r.e. Since $\{A\}$ is Π^0_2, $\deg_w(\{A\} \cup \mathsf{PA}) \in \mathcal{P}_w$. This gives our embedding of $\mathcal{E}_T$ into $\mathcal{P}_w$.

The Embedding Lemma (restated)**:**

Let $S \subseteq \omega^\omega$ be Σ^0_3. Let $P \subseteq 2^\omega$ be nonempty Π^0_1. Then $\exists$ nonempty Π^0_1 $Q \subseteq 2^\omega$ such that $Q \equiv_w S \cup P$.

Proof (sketch). **Step 1.** By Skolem functions, we may assume that $S \subseteq \omega^\omega$ is Π^0_1.

Step 2. We have $S = \{\text{paths through } T_S\}$, $P = \{\text{paths through } T_P\}$, where T_S, T_P are recursive subtrees of $\omega^{<\omega}$, $2^{<\omega}$ respectively. May assume $\tau(n) \geq 2$ for all $n < |\tau|$, $\tau \in T_S$.

Define $Q = \{\text{paths through } T_Q\}$, where T_Q is the set of all $\rho \in \omega^{<\omega}$ of the form

$$\rho = \sigma_0{}^\frown\langle m_0\rangle^\frown\sigma_1{}^\frown\langle m_1\rangle^\frown\cdots{}^\frown\langle m_{k-1}\rangle^\frown\sigma_k$$

where

- $\sigma_0, \sigma_1, \ldots, \sigma_k \in T_P$,
- $\langle m_0, m_1, \ldots, m_{k-1}\rangle \in T_S$,
- $\rho(n) \leq \max(n, 2)$ for all $n < |\rho|$.

One can show that $Q \equiv_w S \cup P$.

Step 3. Q is Π^0_1 and recursively bounded. Hence, we can find Π^0_1 $Q^* \subseteq 2^\omega$ such that Q^* is recursively homeomorphic to Q. Done.

Remark. The Embedding Lemma says that if $\mathbf{s}$ is the weak degree (i.e., Muchnik degree) of a Σ^0_3 mass problem, then $\inf(\mathbf{s},\mathbf{1}) \in \mathcal{P}_w$, where $\mathbf{1} = \deg_w(\mathsf{PA})$ is the top degree in $\mathcal{P}_w$. We have seen that this provides a powerful method of producing specific, natural degrees in $\mathcal{P}_w$. We now extend this method.

If $\mathbf{s}$ is the weak degree of a Σ^0_3 set S, define

$$S' = \{X' \mid X \in S\}$$

where X' is the Turing jump of X, and

$$S^* = \{Y \mid (\exists X \in S)\,(\mathsf{BLR}(X) \subseteq \mathsf{BLR}(Y))\}.$$

Then S' and S^* are Σ^0_3. Moreover, the weak degrees of S' and S^* depend only on the weak degree of S. Thus we have an "internal jump operator" within $\mathcal{P}_w$, given by

$$\inf(\mathbf{s}^*,\mathbf{1}) \mapsto \inf(\mathbf{s}^{*\prime *},\mathbf{1}).$$

In essence, the Cole/Simpson embedding of the hyperarithmetical hierarchy into $\mathcal{P}_w$ may be viewed as iterating the "internal jump operator" through the ordinals $< \omega_1^{\mathsf{CK}}$.

Response to Issue 2:

Issue 2 was:

To find a "smallness property" of an infinite Π^0_1 (i.e., co-r.e.) set $A \subseteq \omega$ which insures that the Turing degree of A is $> \mathbf{0}$ and $< \mathbf{0}'$.

We do not know how to do this.

However, in the $\mathcal{P}_w$ context, we have identified several "smallness properties" of a Π^0_1 set $P \subseteq 2^\omega$ which insure that the weak degree of P is $> \mathbf{0}$ and $< \mathbf{0}'$.

One result of this type:

Theorem (Simpson 2002)**:**

Let $\mathbf{p}$ be the weak degree of a Π^0_1 set $P \subseteq 2^\omega$ which is thin and perfect. Then $\mathbf{p}$ is incomparable with $\mathbf{r}_1$. Hence $\mathbf{0} < \mathbf{p} < \mathbf{0}'$.

Background on thin Π^0_1 sets:

Definition:

A Π^0_1 set $P \subseteq 2^\omega$ is said to be *thin* if, for all Π^0_1 sets $Q \subseteq P$, $P \setminus Q$ is Π^0_1.

Thin perfect Π^0_1 subsets of 2^ω have been constructed by means of priority arguments. Much is known about them. For example, any two such sets are automorphic in the lattice of Π^0_1 subsets of 2^ω under inclusion.

(Martin/Pour-El 1970,
Downey/Jockusch/Stob 1990, 1996,
Cholak/Coles/Downey/Hermann 2001)

Some additional "smallness properties":

Let P be a nonempty Π^0_1 subset of 2^ω.

1. P is *small* if there is no recursive function f such that for all n there exist n members of P which differ at level $f(n)$ in the binary tree. (Binns 2003)

Example: Let $A \subseteq \omega$ be hypersimple, and let $A = B_1 \cup B_2$ where B_1, B_2 are r.e. Then $P = \{X \in 2^\omega \mid X \text{ separates } B_1, B_2\}$ is small.

Theorem (Binns)**:**

If P is small, the weak degree of P is $< \mathbf{0}'$.

2. P is *h-small* if there is no recursive, canonically indexed sequence of pairwise disjoint clopen sets D_n, $n \in \omega$, such that $P \cap D_n \neq \emptyset$ for all n. (Simpson 2003)

Theorem (Simpson)**:**

If P is h-small, the weak degree of P is $< \mathbf{0}'$.

Summary of this talk:

There are basic, unresolved issues concerning $\mathcal{E}_T$, the semilattice of recursively enumerable Turing degrees. One of the issues is the lack of specific, natural, r.e. degrees.

We embed $\mathcal{E}_T$ into $\mathcal{P}_w$, the lattice of weak degrees of nonempty Π^0_1 subsets of 2^ω. We identify $\mathcal{E}_T$ with its image in $\mathcal{P}_w$.

In the $\mathcal{P}_w$ context, some of the unresolved issues can be satisfactorily addressed.

In particular, $\mathcal{P}_w$ contains many specific, natural degrees which are related to foundationally interesting topics:

- algorithmic randomness,
- reverse mathematics,
- computational complexity.

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Some of my papers are available at
`http://www.math.psu.edu/simpson/papers/`.

Transparencies for my talks are available at
`http://www.math.psu.edu/simpson/talks/`.

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