Tutorial on Mass Problems

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Abstract

Let P be a set of reals viewed as a *mass problem*, i.e., a "decision problem with more than one solution." Here the "solutions" of P are the elements of P. Many unsolvable mathematical problems are best viewed as mass problems. One says that a mass problem P is *weakly reducible* to a mass problem Q if for every solution Y of Q there exists a solution X of P such that X is Turing reducible to Y. A *weak degree* is an equivalence class of mass problems under mutual weak reducibility. Weak degrees are also known as *Muchnik degrees*.

Let $\mathcal{P}_w$ be the set of weak degrees of mass problems associated with nonempty Π^0_1 subsets of 2^ω, partially ordered by weak reducibility. Algebraically, it is easy to see that $\mathcal{P}_w$ is a countable distributive lattice with top and bottom. Simpson and others have studied the lattice $\mathcal{P}_w$ in a series of publications beginning in 1999. Our principal findings are as follows.

1. There is a natural embedding of $\mathcal{E}_T$, the countable semilattice of recursively enumerable Turing degrees, into the lattice $\mathcal{P}_w$. This embedding is one-to-one and preserves the semilattice structure and the top and bottom. We identify $\mathcal{E}_T$ with its image in $\mathcal{P}_w$ under this embedding.

2. Like the semilattice $\mathcal{E}_T$, the lattice $\mathcal{P}_w$ is structurally rich. In particular, any countable distributive lattice is lattice-embeddable in any nontrivial initial segment of $\mathcal{P}_w$. Moreover, the $\mathcal{P}_w$ analog of the Sacks Splitting Theorem holds. These structural results are proved by means of priority arguments. The $\mathcal{P}_w$ analog of the Sacks Density Theorem remains as an open problem.

3. Unlike $\mathcal{E}_T$, the lattice $\mathcal{P}_w$ contains a large number of specific, natural degrees other than the top and bottom degrees. These specific, natural degrees in $\mathcal{P}_w$ arise from foundationally interesting topics such as reverse mathematics, algorithmic randomness, subrecursive hierarchies, computational complexity, and hyperarithmeticity.

4. The known specific, natural degrees in $\mathcal{P}_w$ are disjoint from the recursively enumerable Turing degrees in $\mathcal{E}_T$. The only exceptions are the top and bottom degrees in $\mathcal{E}_T$, which are the same as the top and bottom degrees in $\mathcal{P}_w$.

References

[1] Stephen Binns. A splitting theorem for the Medvedev and Muchnik lattices. *Mathematical Logic Quarterly*, 49:327–335, 2003.

[2] Stephen Binns and Stephen G. Simpson. Embeddings into the Medvedev and Muchnik lattices of Π^0_1 classes. *Archive for Mathematical Logic*, 43:399–414, 2004.

[3] C.-T. Chong, Q. Feng, T. A. Slaman, W. H. Woodin, and Y. Yang, editors. *Computational Prospects of Infinity: Proceedings of the Logic Workshop at the Institute for Mathematical Sciences, June 20 – August 15, 2005.* Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore. World Scientific Publishing Company, Ltd. In preparation, 2006, to appear.

[4] Joshua Cole and Stephen G. Simpson. Mass problems and hyperarithmeticity. In preparation, 2006, to appear.

[5] FOM e-mail list. `http://www.cs.nyu.edu/mailman/listinfo/fom/`, September 1997 to the present.

[6] A. A. Muchnik. On strong and weak reducibilities of algorithmic problems. *Sibirskii Matematicheskii Zhurnal*, 4:1328–1341, 1963. In Russian.

[7] Stephen G. Simpson. FOM: natural r.e. degrees; Pi01 classes. FOM e-mail list [5], August 13, 1999.

[8] Stephen G. Simpson. Mass problems and randomness. *Bulletin of Symbolic Logic*, 11:1–27, 2005.

[9] Stephen G. Simpson. An extension of the recursively enumerable Turing degrees. *Journal of the London Mathematical Society*. Preprint, 15 pages, August 10, 2004; accepted for publication July 4, 2006.

[10] Stephen G. Simpson. FOM: natural r.e. degrees. FOM e-mail list [5], February 27, 2005.

[11] Stephen G. Simpson. Some fundamental issues concerning degrees of unsolvability. In [3]. Preprint, 21 pages, 2006, to appear.

[12] Stephen G. Simpson. Mass problems and almost everywhere domination. In preparation, 2006, to appear.