The Gödel Hierarchy and Reverse Mathematics

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Of the 23 Hilbert problems, 1 and 2 belong to foundations of mathematics, while 10 and 17 are closely related to mathematical logic. In addition, problems 3, 4, 5 were an outgrowth of Hilbert's interest in foundations of geometry.

Hilbert's 1900 Problem List

1. Cantor's Problem of the Cardinal Number of the Continuum.

2. Compatibility of the Arithmetical Axioms.

. . .

10. Determination of the Solvability of a Diophantine Equation.

. . .

17. Expression of Definite Forms by Squares.

. . .

Gödel's Incompleteness Theorem:

Hilbert's concern for consistency proofs led to Gödel's Incompleteness Theorems.

Let T be a theory in the predicate calculus, satisfying certain mild conditions. Then:

1. T is incomplete.

2. The statement "T is consistent" is not a theorem of T.

(Gödel 1931)

3. The problem of deciding whether a given formula is a theorem of T is algorithmically unsolvable.

(Gödel, Turing, Tarski, . . .)

Some people believe that Gödel's Incompleteness Theorem means the end of the axiomatic method generally, and of Hilbert's Program specifically.

In my opinion, this view fails to take account of f.o.m. developments subsequent to 1931. The purpose of this talk is to outline some relatively recent research which reveals logical regularity and structure arising from the axiomatic approach to f.o.m.

1. The Gödel Hierarchy

2. Reverse Mathematics

3. Foundational consequences of R. M.

4. A partial realization of Hilbert's Program

5. Beyond the Big Five

6. Relationship to degrees of unsolvability

The Gödel Hierarchy:

Let T_1, T_2 be two theories as above. Define

$$T_1 < T_2$$

if "T_1 is consistent" is a theorem of T_2.

Usually this is equivalent to saying that T_1 is interpretable in T_2 and not vice versa.

This ordering gives a hierarchy of foundational theories, the Gödel Hierarchy.

The Gödel Hierarchy is linear and exhibits other remarkable regularities.

The Gödel Hierarchy is a central object of study in foundations of mathematics.

Stopping Points in the Gödel Hierarchy:

strong:

$\vdots$
supercompact cardinal
$\vdots$
measurable cardinal
$\vdots$
ZFC (ZF set theory with choice)
Zermelo set theory
simple type theory

medium:

Z_2 (2nd order arithmetic)
$\vdots$
Π^1_2 comprehension
Π^1_1 comprehension
ATR_0 (arith. transfinite recursion)
ACA_0 (arithmetical comprehension)

weak:

WKL_0 (weak König's lemma)
RCA_0 (recursive comprehension)
PRA (primitive recursive arithmetic)
EFA (elementary arithmetic)
bounded arithmetic
$\vdots$

Foundations of mathematics (f.o.m.):

Foundations of mathematics is the study of the most basic concepts and logical structure of mathematics as a whole.

Among the most basic mathematical concepts are: number, shape, set, function, algorithm, mathematical proof, mathematical definition, mathematical axiom, mathematical theorem, mathematical statement.

A key f.o.m. question:

What are the appropriate axioms for mathematics?

. . .

Background of Reverse Mathematics:

Second-order arithmetic (Z_2) is a two-sorted system.

Number variables $m, n, \ldots$ range over

$$\mathbb{N} = \omega = \{0, 1, 2, \ldots\} .$$

Set variables $X, Y, \ldots$ range over subsets of ω.

We have $+$, $\times$, $=$ on ω, plus the membership relation

$$\in \quad = \quad \{(n, X) : n \in X\} \quad \subseteq \quad \omega \times P(\omega) .$$

Within subsystems of second-order arithmetic, we can formalize rigorous mathematics (analysis, algebra, geometry, combinatorics).

Subsystems of second-order arithmetic are basic to our current understanding of the logical structure of contemporary mathematics.

An essential reference:

David Hilbert and Paul Bernays
Grundlagen der Mathematik
("Foundations of Mathematics")
Second Edition
Volume I, XV + 475 pages
Volume II, XIV + 561 pages
Grundlehren der Mathematischen Wißenschaften
Springer-Verlag, 1968–1970

In Supplement IV ("Appendix IV") of *Grundlagen der Mathematik*, Hilbert and Bernays present the formalization of rigorous mathematics within second-order arithmetic ($= Z_2$).

Themes of Reverse Mathematics:

Reverse Mathematics is particular f.o.m. research program.

Let τ be a mathematical theorem. Let S_τ be the weakest natural subsystem of second-order arithmetic in which τ is provable.

1. Very often, the principal axiom of S_τ is logically equivalent to τ.

2. Furthermore, only a few subsystems of second-order arithmetic arise in this way.

For a full exposition, see my book.

Two books on reverse mathematics, a status report:

1. RM2001

S. G. Simpson, editor
Reverse Mathematics 2001
(a volume of papers by various authors)
Volume 21, Lecture Notes in Logic
Association for Symbolic Logic
VIII + 401 pages, 2005

2. SOSOA

Stephen G. Simpson
Subsystems of Second-Order Arithmetic
Second Printing
Perspectives in Logic
Association for Symbolic Logic
approximately 460 pages, in press

The Big Five:

For Reverse Mathematics, the five most important subsystems of Z_2 are:

RCA_0 = formalized computable mathematics (Recursive Comprehension Axiom)

$\mathsf{WKL}_0 = \mathsf{RCA}_0 +$ a compactness principle (Weak König's Lemma)

$\mathsf{ACA}_0 = \mathsf{RCA}_0 +$ the Turing jump operator (Arithmetical Comprehension Axiom)

$\mathsf{ATR}_0 = \mathsf{ACA}_0 +$ transfinite recursion (Arithmetical Transfinite Recursion)

$\Pi^1_1\text{-}\mathsf{CA}_0 = \mathsf{ACA}_0 + \Pi^1_1$ comprehension (Π^1_1 Comprehension Axiom)

Themes of R. M. (continued):

We develop a table indicating which mathematical theorems can be proved in which subsystems of Z_2.

	RCA_0	WKL_0	ACA_0	ATR_0	$\Pi^1_1\text{-}\mathsf{CA}_0$
analysis (separable):					
differential equations	X	X			
continuous functions	X, X	X, X	X		
completeness, *etc.*	X	X	X		
Banach spaces	X	X, X			X
open and closed sets	X	X		X, X	X
Borel and analytic sets	X			X, X	X, X
algebra (countable):					
countable fields	X	X, X	X		
commutative rings	X	X	X		
vector spaces	X		X		
Abelian groups	X		X	X	X
miscellaneous:					
mathematical logic	X	X			
countable ordinals	X		X	X, X	
infinite matchings		X	X	X	
the Ramsey property			X	X	X
infinite games			X	X	X

Reverse Mathematics for WKL_0

WKL_0 is equivalent over RCA_0 to each of the following mathematical statements:

1.The Heine/Borel Covering Lemma: Every covering of $[0, 1]$ by a sequence of open intervals has a finite subcovering.

2. Every covering of a compact metric space by a sequence of open sets has a finite subcovering.

3. Every continuous real-valued function on $[0, 1]$ (or on any compact metric space) is bounded (uniformly continuous, Riemann integrable).

6. The Maximum Principle: Every continuous real-valued function on $[0, 1]$ (or on any compact metric space) has (or attains) a supremum.

7. The local existence theorem for solutions of (finite systems of) ordinary differential equations.

8. Gödel's Completeness Theorem: every finite (or countable) set of sentences in the predicate calculus has a countable model.

9. Every countable commutative ring has a prime ideal.

10. Every countable field (of characteristic 0) has a unique algebraic closure.

11. Every countable formally real field is orderable.

12. Every countable formally real field has a (unique) real closure.

13. Brouwer's Fixed Point Theorem: Every (uniformly) continuous function $\phi : [0, 1]^n \to [0, 1]^n$ has a fixed point.

Reverse Mathematics for WKL_0 (continued)

14. The Separable Hahn/Banach Theorem: If f is a bounded linear functional on a subspace of a separable Banach space, and if $\|f\| \leq 1$, then f has an extension $\widetilde{f}$ to the whole space such that $\|\widetilde{f}\| \leq 1$.

15. Banach's Theorem: In a separable Banach space, given two disjoint convex open sets A and B, there exists a closed hyperplane H such that A is on one side of H and B is on the other.

16. Every countable k-regular bipartite graph has a perfect matching.

Reverse Mathematics for ACA_0

ACA_0 is equivalent over RCA_0 to each of the following mathematical statements:

1. Every bounded, or bounded increasing, sequence of real numbers has a least upper bound.

2. The Bolzano/Weierstraß Theorem: Every bounded sequence of real numbers, or of points in $\mathbb{R}^n$, has a convergent subsequence.

3. Every sequence of points in a compact metric space has a convergent subsequence.

4. The Ascoli Lemma: Every bounded equicontinuous sequence of real-valued continuous functions on a bounded interval has a uniformly convergent subsequence.

5. Every countable commutative ring has a maximal ideal.

6. Every countable vector space over $\mathbb{Q}$, or over any countable field, has a basis.

7. Every countable field (of characteristic 0) has a transcendence basis.

8. Every countable Abelian group has a unique divisible closure.

9. König's Lemma: Every infinite, finitely branching tree has an infinite path.

10. Ramsey's Theorem for colorings of $[\mathbb{N}]^3$, or of $[\mathbb{N}]^4$, $[\mathbb{N}]^5$, ...

Reverse Mathematics for ATR_0

ATR_0 is equivalent over RCA_0 to each of the following mathematical statements:

1. Any two countable well orderings are comparable.

2. Ulm's Theorem: Any two countable reduced Abelian p-groups which have the same Ulm invariants are isomorphic.

3. The Perfect Set Theorem: Every uncountable closed, or analytic, set has a perfect subset.

4. Lusin's Separation Theorem: Any two disjoint analytic sets can be separated by a Borel set.

5. The domain of any single-valued Borel set in the plane is a Borel set.

6. Every clopen (or open) game in $\mathbb{N}^\mathbb{N}$ is determined.

7. Every clopen (or open) subset of $[\mathbb{N}]^\mathbb{N}$ has the Ramsey property.

8. Every countable bipartite graph admits a König covering.

Reverse Mathematics for Π^1_1-CA_0

Π^1_1-CA_0 is equivalent over RCA_0 to each of the following mathematical statements:

1. Every tree has a largest perfect subtree.

2. The Cantor/Bendixson Theorem: Every closed subset of $\mathbb{R}$, or of any complete separable metric space, is the union of a countable set and a perfect set.

3. Every countable Abelian group is the direct sum of a divisible group and a reduced group.

4. Every difference of two open sets in the Baire space $\mathbb{N}^{\mathbb{N}}$ is determined.

5. Every G_δ set in $[\mathbb{N}]^{\mathbb{N}}$ has the Ramsey property.

6. Silver's Theorem: For every Borel (or coanalytic, or F_σ) equivalence relation with uncountably many equivalence classes, there exists a perfect set of inequivalent elements.

7. For every countable set S in the dual X^* of a separable Banach space X (or in $\ell_1 = c_0^*$), there exists a smallest weak-$*$-closed subspace of X^* (or of ℓ_1) containing S.

8. For every norm-closed subspace Y of $\ell_1 = c_0^*$, the weak-$*$-closure of Y exists.

My Reverse Mathematics Ph.D. students:

1. John Steel, *Determinateness and Subsystems of Analysis*, University of California at Berkeley, 1977.

2. Rick L. Smith, *Theory of Profinite Groups with Effective Presentations*, Pennsylvania State University, 1979.

5. Stephen H. Brackin, *On Ramsey-type Theorems and their Provability in Weak Formal Systems*, Pennsylvania State University, 1984.

7. Douglas K. Brown, *Functional Analysis in Weak Subsystems of Second Order Arithmetic*, Pennsylvania State University, 1987.

8. Jeffry L. Hirst, *Combinatorics in Subsystems of Second Order Arithmetic*, Pennsylvania State University, 1987.

9. Xiaokang Yu (Connie Yu), *Measure Theory in Weak Subsystems of Second Order Arithmetic*, Pennsylvania State University, 1987.

10. Fernando Ferreira, *Polynomial Time Computable Arithmetic and Conservative Extensions*, Pennsylvania State University, 1988.

11. Kostas Hatzikiriakou, *Commutative Algebra in Subsystems of Second Order Arithmetic*, Pennsylvania State University, 1989.

12. Alberto Marcone, *Foundations of BQO Theory and Subsystems of Second Order Arithmetic*, Pennsylvania State University, 1993.

13. A. James Humphreys, *On the Necessary Use of Strong Set Existence Axioms in Analysis and Functional Analysis*, Pennsylvania State University, 1996.

14. Mariagnese Giusto, *Topology, Analysis and Reverse Mathematics*, University of Torino, 1998.

16. Carl Mummert, *On the Reverse Mathematics of General Topology*, Pennsylvania State University, 2005.

Foundational consequences of R. M.:

1. We precisely classify mathematical theorems, according to which subsystems of Z_2 they are provable in.

2. We identify certain subsystems of Z_2 as being mathematically natural.

The naturalness is rigorously demonstrated.

3. We work out the consequences of particular foundational doctrines:

- recursive analysis (Pour-El/Richards)
- constructivism (Bishop)
- finitistic reductionism (Hilbert)
- predicativity (Weyl, Feferman)
- predicative reductionism (Feferman, Friedman, Simpson)
- impredicative analysis (Takeuti, Schütte, Pohlers)

Foundational consequences (continued):

By means of Reverse Mathematics, we identify five particular subsystems of Z_2 as being mathematically natural. We correlate these systems to traditional f.o.m. programs.

RCA_0	constructivism	Bishop
WKL_0	finitistic reductionism	Hilbert
ACA_0	predicativity	Weyl, Feferman
ATR_0	predicative reductionism	Friedman, Simpson
Π^1_1-CA_0	impredicativity	Feferman *et al.*

We analyze these f.o.m. programs in terms of their consequences for mathematical practice. Specifically, under the various proposals, which mathematical theorems are "lost"? Reverse Mathematics provides precise answers to such questions.

Hilbert's Program:

Hilbert 1925 proposed to reduce all of mathematics to finitistic mathematics.

Gödel's Theorem implies that Hilbert's Program cannot be completely realized. For instance, "first-order arithmetic is consistent" is finitistically meaningful yet not finitistically provable.

Nevertheless, a significant partial realization of Hilbert's Program has been obtained.

Reference:

Stephen G. Simpson, Partial realizations of Hilbert's Program, *Journal of Symbolic Logic*, 53, 1988, 349–363.

See also the next slide.

A partial realization of Hilbert's Program:

1. W. W. Tait has argued that PRA embodies finitism.

2. H. Friedman has shown that WKL_0 is conservative over PRA for Π^0_2 sentences. This class includes all finitistically meaningful sentences.

3. A large portion of core mathematics can be carried out in WKL_0, including many of the best known nonconstructive theorems.

Thus Hilbert's Program can be realized for a large portion of core mathematics.

This is a byproduct of Reverse Mathematics.

Reference:

Stephen G. Simpson, Partial realizations of Hilbert's Program, *Journal of Symbolic Logic*, 53, 1988, 349–363.

Beyond the Big Five:

Mummert/Simpson 2005 provide an example of Reverse Mathematics at the level of Π^1_2 comprehension. The results are in the area of general topology.

Definitions.

Let P be a poset. A *filter* is a set $F \subseteq P$ such that F is upward closed and for all $p, q \in F$ there exists $r \in F$ such that $r \leq p$ and $r \leq q$.

(Compare forcing in axiomatic set theory.)

A *maximal filter* is a filter not properly included in any other filter.

MF(P) is the space of all maximal filters, with topology generated by basic open neighborhoods $N_p = \{F \mid p \in F\}$, $p \in P$.

An MF-*space* is a topological space of the form $X =$ MF(P). If P is countable, we say that X is *countably-based*.

MF-spaces (continued):

The MF-spaces comprise a wide class of topological spaces, including all complete metric spaces and many non-metrizable spaces. In particular, every Polish space is a countably-based MF-space.

Consider the following statement, MFMT:

> A countably-based MF-space is regular if and only if it is completely metrizable.

Note that MFMT can be formalized as a sentence in the language of second-order arithmetic.

Theorem (Mummert/Simpson 2005).
MFMT is equivalent to Π^1_2-CA_0 over Π^1_1-CA_0.

This is the first (and so far the only) example of Reverse Mathematics at the level of Π^1_2 comprehension.

Beyond the Big Five (continued):

Other results and problems concern systems in the vicinity of WKL_0.

We mention some results in the Reverse Mathematics of measure theory.

Definition (Xiaokang Yu, 1986).

WWKL_0 (*weak weak König's Lemma*) consists of RCA_0 plus the following statement:

If T is a subtree of $2^{<\mathbb{N}}$ and $\lim_{n\to\infty} \dfrac{|T \cap 2^n|}{2^n} > 0$,

then T has an infinite path.

It can be shown that $\mathsf{RCA}_0 \subsetneqq \mathsf{WWKL}_0 \subsetneqq \mathsf{WKL}_0$.

Many statements of measure theory are equivalent to WWKL_0 over RCA_0. Examples:

- finite additivity
- countable additivity
- the Monotone Convergence Theorem
- the Vitali Covering Lemma.

Reverse Mathematics of measure theory (continued):

It can be shown that WWKL_0 is equivalent over RCA_0 to

$$\forall X\, \exists Y\, (Y \text{ is Martin-Löf random rel. to } X).$$

Thus we have a good tie-in to recent work on algorithmic randomness.

A second wave of research in the Reverse Mathematics of measure theory concerns *almost everywhere domination*.

Definition (Dobrinen/Simpson 2004).

A Turing oracle B is said to be a.e. dominating if for almost all $X \in 2^{\mathbb{N}}$ and all $f \leq_T X$ there exists $g \leq_T B$ such that $f(n) < g(n)$ for all n.

Here "almost all" refers to the fair coin measure on $2^{\mathbb{N}}$.

Almost everywhere domination (continued):

Almost everywhere domination is closely related to the reverse mathematics of measure-theoretic regularity. For instance, B is a.e. dominating if and only if every Π^0_2 set of reals contains a $\Sigma^{0,B}_2$ set of the same measure. See also recent definitive results by Kjos-Hanssen, Miller, Solomon.

It turns out that B is a.e. dominating if and only if the Halting Problem is LR-*reducible to* B in the sense of André Nies.

The relevant definition is: $A \leq_{\mathsf{LR}} B$ if

$\forall X\,(X$ random rel. to $B \Rightarrow X$ random rel. to $A)$.

Thus we have another good tie-in between the Reverse Mathematics of measure theory and recent research in algorithmic randomness.

Degrees of unsolvability:

There are close connections between subsystems of Z_2 and degrees of unsolvability.

RCA_0 corresponds to the Turing degree $\mathbf{0}$.

ACA_0 corresponds to the Turing degree $\mathbf{0}'$ or perhaps $\mathbf{0}^{(\omega)}$.

Π^1_1-CA_0 corresponds to $\mathbf{0}^{\left(\omega_1^{\mathrm{CK}}\right)}$ (= the Turing degree of Kleene's O) or perhaps $\mathbf{0}^{\left(\omega_\omega^{\mathrm{CK}}\right)}$.

Similarly for Π^1_2-CA_0, etc.

However, it is difficult to assign a Turing degree to systems such as WKL_0, WWKL_0, ATR_0, etc.

To overcome such difficulties, we propose to use a generalization of Turing degrees known as *Muchnik degrees*.

Muchnik degrees (continued):

Definition. Let P and Q be sets of reals. P *is Muchnik reducible to* Q, abbreviated $P \leq_w Q$, if $(\forall y \in Q)\,(\exists x \in P)\,(x \leq_T y)$.

The *Muchnik degree of* P, denoted $\deg_w(P)$, is the equivalence class of P under mutual Muchnik reducibility.

To motivate this concept, view P as a *mass problem*, namely, the "problem" of finding an element of P.

Then $P \leq_w Q$ means: any "solution" of the problem Q can be used to compute a "solution" of the problem P.

The Muchnik degree corresponding to WKL_0 is $\deg_w(\mathsf{CPA})$ where CPA is the problem of finding a completion of Peano Arithmetic.

Similarly, the Muchnik degree corresponding to WWKL_0 is $\mathbf{r}_1 = \deg_w(\mathsf{R}_1)$ where R_1 is the problem of finding a 1-random real.

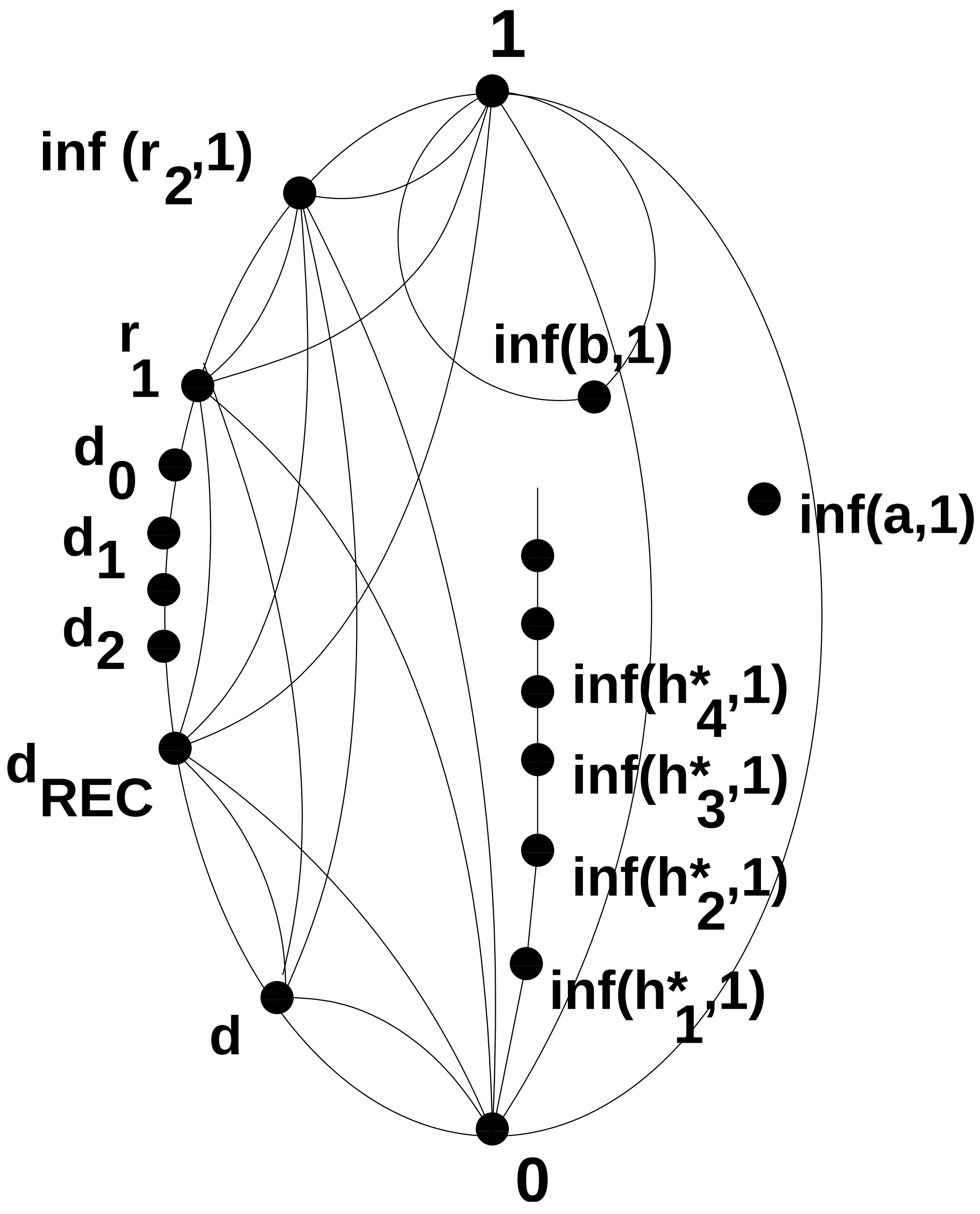
Muchnik degrees (continued):

We have been studying the lattice $\mathcal{P}_w$ of Muchnik degrees of nonempty Π^0_1 sets of reals. It turns out that $\mathcal{P}_w$ is structurally similar to the r.e. Turing degrees, but much better than the r.e. Turing degrees because:

1. $\mathcal{P}_w$ is a distributive lattice, while the r.e. Turing degrees are not even a lattice.

2. $\mathcal{P}_w$ contains many specific, natural degrees which are closely related to foundationally interesting topics:

- algorithmic randomness
- reverse mathematics
- almost everywhere domination
- diagonal nonrecursiveness
- hyperarithmeticity
- resource-bounded computational complexity
- Kolmogorov complexity
- effective Hausdorff dimension
- subrecursive hierarchies



A picture of $\mathcal{P}_w$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{h} =$ hyperarithmeticity, $\mathbf{r} =$ randomness, $\mathbf{b} =$ almost everywhere domination, $\mathbf{d} =$ diagonal nonrecursiveness.

Similarly, the lattice $\mathcal{D}_w$ of all Muchnik degrees is better behaved and has many more specific, natural degrees than the upper semilattice of Turing degrees. In addition, there are important connections to intuitionism going back to Kolmogorov.

Some specific, natural degrees in $\mathcal{D}_w$:

- $\mathbf{0}, \mathbf{0}', \mathbf{0}'', \ldots, \mathbf{0}^{(\alpha)}, \ldots$ where α runs through the constructibly countable ordinals (at least, depending on the universe of set theory).

- $\mathbf{r}_n = \deg_w(\mathsf{R}_n)$ where R_n is the problem of finding an n-random real.

- $\mathbf{b} = \deg_w(\mathsf{AED})$ where AED is the problem of finding an a.e. dominating real.

- $\mathbf{c}_2 = \deg_w(\mathsf{C}_2)$ where C_2 is the problem of finding a *Ramsey real*, i.e., a real which computes homogenous sets for all recursive colorings of pairs.

- $\cdots$

THE END