Aspects of the Muchnik lattice

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Turing degrees and Muchnik degrees.

A *decision problem* is a real $X \in \mathbb{N}^{\mathbb{N}}$. Intuitively, X represents the problem of "finding" or "computing" X. For $X, Y \in \mathbb{N}^{\mathbb{N}}$ we say that *X is Turing reducible to Y*, abbreviated $X \leq_{\mathsf{T}} Y$, if X is computable using Y as a Turing oracle. A *Turing degree* is an equivalence class of decision problems under mutual Turing reducibility. The Turing degree of X is denoted $\deg_{\mathsf{T}}(X)$. The partial ordering of all Turing degrees is denoted $\mathcal{D}_{\mathsf{T}}$. $\mathcal{D}_{\mathsf{T}}$ is not a lattice, but it is an upper semilattice.

A *mass problem* is a subset of $\mathbb{N}^{\mathbb{N}}$. Intuitively, $P \subseteq \mathbb{N}^{\mathbb{N}}$ is identified with the problem of "finding" or "computing" at least one member of P. Any $X \in P$ is a "solution" of this problem.

For $P, Q \subseteq \mathbb{N}^{\mathbb{N}}$ we say that *P is Muchnik reducible to Q*, abbreviated $P \leq_{\mathsf{w}} Q$, if $\forall Y\,(Y \in Q \Rightarrow \exists X\,(X \in P \text{ and } X \leq_{\mathsf{T}} Y))$. Intuitively, from any "solution" of Q we can compute a "solution" of P.

A *Muchnik degree* is an equivalence class of mass problems under mutual Muchnik reducibility. The Muchnik degree of P is denoted $\deg_{\mathsf{w}}(P)$. The partial ordering of all Muchnik degrees is denoted $\mathcal{D}_{\mathsf{w}}$.

Turing degrees and Muchnik degrees (continued).

Identifying $X \in \mathbb{N}^{\mathbb{N}}$ with $\{X\} \subseteq \mathbb{N}^{\mathbb{N}}$, we have an embedding $\mathcal{D}_{\mathsf{T}} \hookrightarrow \mathcal{D}_{\mathsf{w}}$ which induces a one-to-one order-reversing correspondence between $\mathcal{D}_{\mathsf{w}}$ and upwardly closed subsets of $\mathcal{D}_{\mathsf{T}}$. Thus $\mathcal{D}_{\mathsf{w}} = \widehat{\mathcal{D}_{\mathsf{T}}} =$ the *completion* of $\mathcal{D}_{\mathsf{T}}$. It follows that $\mathcal{D}_{\mathsf{w}}$ is a complete and completely distributive lattice.

We may view $\mathcal{D}_{\mathsf{T}}$ as a topological space in which the open sets are the upwardly closed sets. Thus $\mathcal{D}_{\mathsf{w}}$ is a topological model of intuitionistic propositional calculus.

For any topological space $\mathcal{T}$, a *sheaf* over $\mathcal{T}$ consists of a topological space $\mathcal{X}$ together with a local homeomorphism $p : \mathcal{X} \to \mathcal{T}$. A *sheaf morphism* from a sheaf $p : \mathcal{X} \to \mathcal{T}$ to another sheaf $q : \mathcal{Y} \to \mathcal{T}$ is a continuous function $f : \mathcal{X} \to \mathcal{Y}$ such that $p(x) = q(f(x))$ for all $x \in \mathcal{X}$.
Let $\mathsf{Sh}(\mathcal{T}) =$ the category of sheaves and sheaf morphisms over $\mathcal{T}$.
$\mathsf{Sh}(\mathcal{T})$ is a topos and a model of intuitionistic higher-order logic.
In this model, the truth values are the open subsets of $\mathcal{T}$.

The Muchnik topos and the Muchnik reals.

Applying the above construction to the topological space $\mathcal{D}_{\mathrm{T}}$, we obtain $\mathrm{Sh}(\mathcal{D}_{\mathrm{T}})$ = the *Muchnik topos*. In this model of intuitionistic mathematics, the truth values are the Muchnik degrees. We offer $\mathrm{Sh}(\mathcal{D}_{\mathrm{T}})$ as a rigorous implementation of Kolmogorov's nonrigorous 1932 interpretation of intuitionistic mathematics as a "calculus of problems."

Consider the topological space $\mathbb{R}_C = \mathbb{R} \times \mathcal{D}_{\mathrm{T}}$ with basic open sets $\{x\} \times \mathcal{U}$ where $x \in \mathbb{R}$ and $\mathcal{U} \subseteq \mathcal{D}_{\mathrm{T}}$ is upwardly closed. There is a projection map $p : \mathbb{R}_C \to \mathcal{D}_{\mathrm{T}}$ given by $p(x, \mathbf{a}) = \mathbf{a}$. Thus $\mathbb{R}_C$ is a sheaf over $\mathcal{D}_{\mathrm{T}}$ representing the Cauchy/Dedekind real number system. An interesting subsheaf of $\mathbb{R}_C$ is $\mathbb{R}_M = \{(x, \mathbf{a}) \in \mathbb{R}_C \mid \mathrm{deg}_{\mathrm{T}}(x) \leq \mathbf{a}\}$, the sheaf of *Muchnik reals*, which supports an analog of computable analysis. Intuitively, a Cauchy/Dedekind real can exist anywhere within the Turing degrees, but a Muchnik real can exist only where we have enough Turing oracle power to compute it. We have proved that the Muchnik topos satisfies a *Choice and Bounding Principle*

$$(\forall x\, \exists y\, \Phi(x, y)) \Rightarrow (\exists w\, \exists z\, \forall x\, (wx \leq_{\mathrm{T}} (x, z) \text{ and } \Phi(x, wx)))$$

where x, y, z range over Muchnik reals, w ranges over functions from Muchnik reals to Muchnik reals, and w and z do not occur in Φ.

The sublattices $\mathcal{E}_{\mathsf{w}}$ and $\mathcal{S}_{\mathsf{w}}$.

Since $\mathcal{D}_{\mathsf{w}}$ is large and complicated, it is natural to consider sublattices which are more manageable. Two such sublattices are

$$\mathcal{E}_{\mathsf{w}} = \{\deg_{\mathsf{w}}(P) \mid \emptyset \neq P \subseteq \{0,1\}^{\mathbb{N}} \text{ and } P \text{ is } \Pi^0_1\}$$

and

$$\mathcal{S}_{\mathsf{w}} = \{\deg_{\mathsf{w}}(P) \mid \emptyset \neq P \subseteq \mathbb{N}^{\mathbb{N}} \text{ and } P \text{ is } \Pi^0_1\}\,.$$

We compare $\mathcal{E}_{\mathsf{w}}$ to $\mathcal{E}_{\mathsf{T}}$ = the upper semilattice of r.e. Turing degrees.

There is a strong analogy between $\mathcal{E}_{\mathsf{w}}$ and $\mathcal{E}_{\mathsf{T}}$:

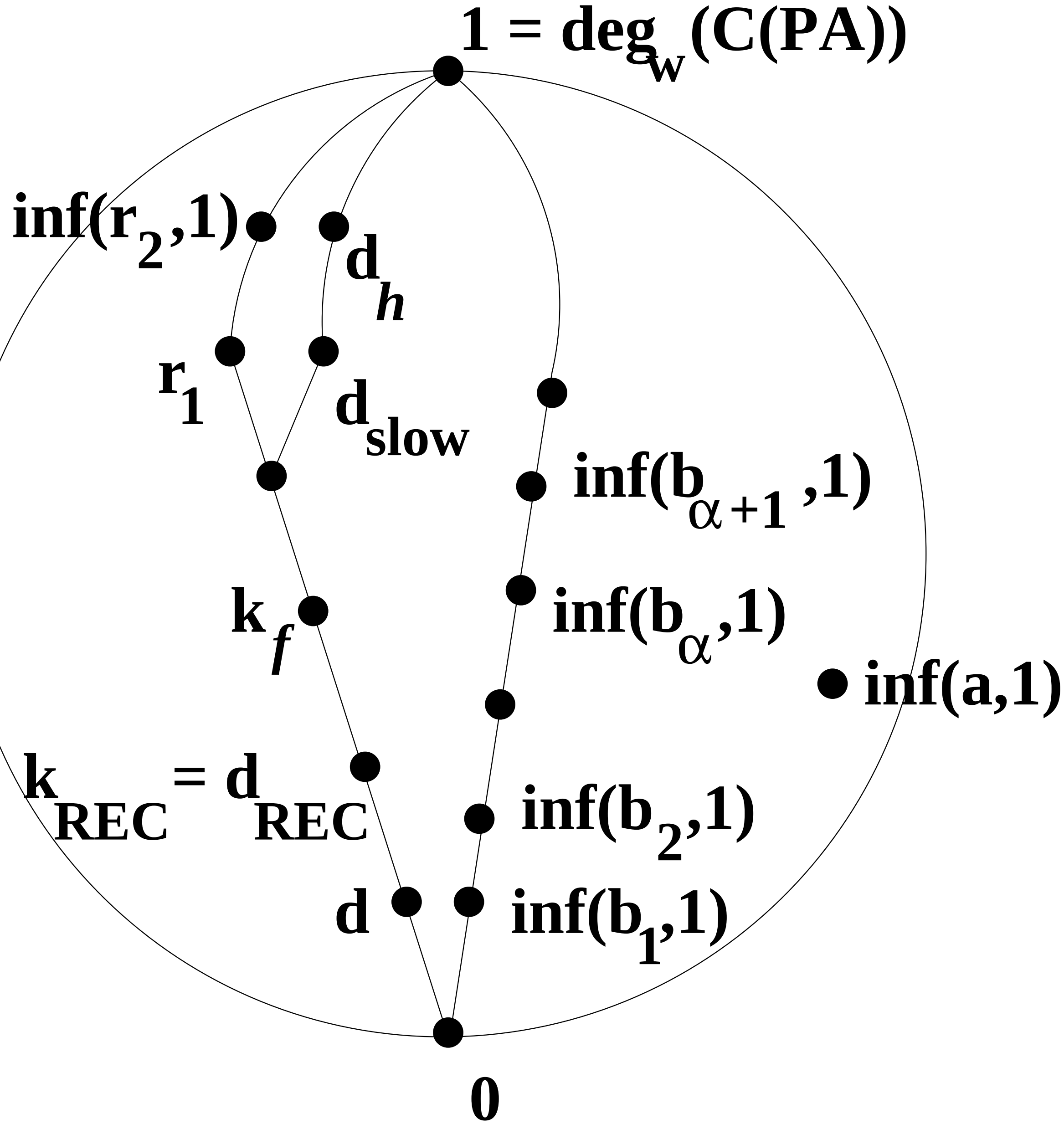
(a) $\mathcal{E}_{\mathsf{w}}$ is the smallest natural sublattice of $\mathcal{D}_{\mathsf{w}}$, just as $\mathcal{E}_{\mathsf{T}}$ is the smallest natural subsemilattice of $\mathcal{D}_{\mathsf{T}}$.

(b) There is a natural embedding $\mathbf{a} \mapsto \inf(\mathbf{a}, \mathbf{1}) : \mathcal{E}_{\mathsf{T}} \hookrightarrow \mathcal{E}_{\mathsf{w}}$.

(c) The Splitting Theorem and the Density Theorem, due to Sacks for $\mathcal{E}_{\mathsf{T}}$, also hold for $\mathcal{E}_{\mathsf{w}}$. See below.

However, $\mathcal{E}_{\mathsf{w}}$ has an advantage over $\mathcal{E}_{\mathsf{T}}$:

$\mathcal{E}_{\mathsf{w}}$ contains many specific, natural degrees associated with specific, natural, foundationally interesting problems. In contrast, $\mathcal{E}_{\mathsf{T}}$ is not known to contain any such degrees other than $\mathbf{0}'$ and $\mathbf{0}$.



This is a picture of $\mathcal{E}_{\mathsf{w}}$. Each black dot except $\mathsf{inf}(\mathbf{a}, 1)$ represents a specific, natural, Muchnik degree in $\mathcal{E}_{\mathsf{w}}$.

The Splitting and Density Theorems for $\mathcal{E}_{\mathsf{w}}$.

Splitting Theorem (Binns 2003). $\mathcal{E}_{\mathsf{w}}$ satisfies the Splitting Theorem: $\forall x\,(x > 0 \Rightarrow \exists u\,\exists v\,(u < x$ and $v < x$ and $x = \sup(u, v)))$.

Density Theorem (Binns/Shore/Simpson 2014). $\mathcal{E}_{\mathsf{w}}$ satisfies the Density Theorem: $\forall x\,\forall y\,(x < y \Rightarrow \exists z\,(x < z < y))$.

We do not know whether $\mathcal{E}_{\mathsf{w}}$ satisfies dense splitting.

We now sketch the proof that $\mathcal{E}_{\mathsf{w}}$ is dense. Since $\mathcal{E}_{\mathsf{w}}$ is an initial segment of $\mathcal{S}_{\mathsf{w}}$ (Simpson 2007), it suffices to prove that $\mathcal{S}_{\mathsf{w}}$ is dense. The proof is presented in a modular way, with several lemmas.

Lemma 1. Let $Q \subseteq \mathbb{N}^{\mathbb{N}}$ be Π^0_1 such that $Q \not\leq_{\mathsf{w}} \{0\}$. Then for all $Y \in \mathbb{N}^{\mathbb{N}}$ there exists $\widehat{Y} \in \mathbb{N}^{\mathbb{N}}$ such that $0' \oplus Y \equiv_{\mathsf{T}} 0' \oplus \widehat{Y} \equiv_{\mathsf{T}} \widehat{Y}'$ and $Q \not\leq_{\mathsf{w}} \{\widehat{Y}\}$.

Lemma 1 is proved like the Friedberg Jump Theorem,
with extra steps taken to insure that $Q \not\leq_{\mathsf{w}} \{\widehat{Y}\}$.

Lemma 2. Given Π^0_1 predicates $U, V \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$, we can find a Π^0_1 predicate $\widehat{U} \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ such that for each X with $\{Z \mid V(X,Z)\} \not\leq_\mathrm{w} \{X\}$ there is a homeomorphism $Y \mapsto \widehat{Y}$ of $\{Y \mid U(X,Y)\}$ onto $\{\widehat{Y} \mid \widehat{U}(X,\widehat{Y})\}$ such that $X' \oplus Y \equiv_\mathrm{T} X' \oplus \widehat{Y} \equiv_\mathrm{T} (X \oplus \widehat{Y})'$ and $\{Z \mid V(X,Z)\} \not\leq_\mathrm{w} \{X \oplus \widehat{Y}\}$.

Lemma 2 is proved by uniformly relativizing Lemma 1 to X, taking extra care to insure that $\{\widehat{Y} \mid \widehat{U}(X,\widehat{Y})\}$ is uniformly Π^0_1 relative to X.

Lemma 3. Suppose Kleene's O is not hyperarithmetical in X. Then, there is a nonempty Π^0_1 set $S \subseteq \mathbb{N}^\mathbb{N}$ such that $S \not\leq_\mathrm{w} \{X'\}$.

Lemma 3 follows from the Kleene Normal Form Theorem plus the fact that Kleene's O is Π^1_1.

We now prove that $\mathcal{S}_\mathrm{w}$ is dense. Given Π^0_1 sets $P, Q \subseteq \mathbb{N}^\mathbb{N}$ such that $P <_\mathrm{w} Q$, to find a Π^0_1 set $R \subseteq \mathbb{N}^\mathbb{N}$ such that $P <_\mathrm{w} R <_\mathrm{w} Q$. By the Gandy Basis Theorem, let $X_0 \in P$ be such that Kleene's O is not hyp. in X_0. By Lemma 3 let $S \subseteq \mathbb{N}^\mathbb{N}$ be nonempty Π^0_1 such that $S \not\leq_\mathrm{w} \{X_0'\}$. Apply Lemma 2 with $U(X,Y) \equiv Y \in S$ and $V(X,Z) \equiv Z \in Q$. Let $R = \{X \oplus \widehat{Y} \mid X \in P \text{ and } \widehat{U}(X,\widehat{Y})\} \cup Q$ where $\widehat{U}$ is as in the conclusion of Lemma 2. It is easy to check that this works.

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Thank you for your attention!