Aspects of the Muchnik lattice

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Let P and Q be sets of reals. Intuitively we may view a set of reals as a "problem," namely, the problem of "finding" some real in the set. Accordingly, we say that P is *Muchnik reducible* to Q if for all $y \in Q$ there exists $x \in P$ such that x is Turing reducible to y. The *Muchnik degree* of P is the equivalence class of P under mutual Muchnik reducibility. Let $\mathcal{D}_{\mathrm{w}}$ be the lattice of all Muchnik degrees, and let $\mathcal{E}_{\mathrm{w}}$ be the sublattice consisting of the Muchnik degrees of nonempty, effectively closed (i.e., Π^0_1) sets of reals. It is well known that $\mathcal{D}_{\mathrm{w}}$ is the natural completion of the upper semilattice $\mathcal{D}_{\mathrm{T}}$ of Turing degrees. Similarly, $\mathcal{E}_{\mathrm{w}}$ is a natural extension of the upper semilattice $\mathcal{E}_{\mathrm{T}}$ of recursively enumerable Turing degrees. In a recent paper by Sankha S. Basu and the speaker, we show that the category of sheaves of sets over $\mathcal{D}_{\mathrm{w}}$ is an interesting model of intuitionistic higher-order logic. We call this model the *Muchnik topos*. In recent work by Stephen E. Binns and Richard A. Shore and the speaker, we show that $\mathcal{E}_{\mathrm{w}}$ is *dense*, i.e., for all $\mathbf{p}, \mathbf{q} \in \mathcal{E}_{\mathrm{w}}$ such that $\mathbf{p} < \mathbf{q}$ there exists $\mathbf{r} \in \mathcal{E}_{\mathrm{w}}$ such that $\mathbf{p} < \mathbf{r} < \mathbf{q}$. We now sketch the proof of this latter result. The proof involves some hyperarithmetical theory.