Symbolic dynamics and degrees of unsolvability

Stephen G. Simpson
Pennsylvania State University
NSF DMS-0600823, DMS-0652637
Templeton Foundation, Grove Endowment
http://www.math.psu.edu/simpson/
simpson@math.psu.edu

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Abstract:

Let A be a finite set of symbols. The 2-*dimensional shift space on* A is $A^{\mathbb{Z}\times\mathbb{Z}}$ with shift operators S_1 and S_2 given by $S_1(x)(m,n)=x(m+1,n)$ and $S_2(x)(m,n)=x(m,n+1)$. A 2-*dimensional subshift* is a nonempty, closed subset of $A^{\mathbb{Z}\times\mathbb{Z}}$ which is invariant under S_1 and S_2. A 2-dimensional subshift is said to be *of finite type* if it is defined by a finite set of excluded finite configurations of symbols. We regard real numbers and points of $A^{\mathbb{Z}\times\mathbb{Z}}$ as Turing oracles. If X and Y are sets of Turing oracles, we say that X *is Muchnik reducible to* Y if each $y \in Y$ can be used to compute some $x \in X$. The *Muchnik degree of* X is the equivalence class of X under mutual Muchnik reducibility. We prove that the Muchnik degrees of 2-dimensional subshifts of finite type are the same as the Muchnik degrees of nonempty, effectively closed sets of real numbers. We then apply known results about such Muchnik degrees to obtain an infinite family of 2-dimensional subshifts of finite type which are, in a certain strong sense, mutually independent. Our application is stated purely in terms of symbolic dynamics, with no mention of Muchnik reducibility.

Announcement

Special Session:
Logic and Dynamical Systems

Sponsors:
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Joint Mathematics Meetings,
Washington DC, January 5–8, 2009

Monday, January 5, 2:15 – 6:10 PM

Tuesday, January 6, 8:00 – 11:55 AM

Virginia Suite B, Lobby Level, Marriott

Organizer: Stephen G. Simpson

Important note for speakers:

Please email me your slides
as soon as possible!

If you cannot email me your slides,
please have them readily available
on a USB memory stick.

This advance preparation will speed things up.

Please remember that the breaks between
talks are only 5 minutes long.

Thank you!

Degrees of unsolvability:

Let $x, y, z, \ldots$ be *Turing oracles*.

Here $x, y, z, \ldots$ can be real numbers ($x \in \mathbb{R}$) or points in Euclidean space ($x \in \mathbb{R}^k$) or number-theoretic functions $x = f : \mathbb{N}^k \to \mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, \ldots\}$, or functions $x : \mathbb{Z} \times \mathbb{Z} \to A$ where A is a finite set of symbols, $A = \{a_1, \ldots, a_k\}$.

We say that x *is Turing reducible to* y, abbreviated $x \leq_T y$, if x is computable by a Turing machine using y as a Turing oracle.

A *mass problem* is a set of Turing oracles. If P and Q are mass problems, we say that P is *Muchnik reducible* or *weakly reducible* to Q, abbreviated $P \leq_w Q$, if for all $y \in Q$ there exists $x \in P$ such that $x \leq_T y$.

Intuitively, viewing P as a "problem", the "solutions" of P are the elements of P. Then, P is said to be "reducible" to Q if every solution of the problem Q can be used as a Turing oracle to compute some solution of the problem P.

Mass problems were introduced by Kolmogorov 1932, Medvedev 1955, Muchnik 1963 as a model of the intuitionistic logic of Brouwer and Heyting.

We are more interested in mass problems from the viewpoint of
degrees of unsolvability.

A mass problem P is said to be *solvable* if it has a computable solution, i.e., there exists a computable x such that $x \in P$. Otherwise P is said to be *unsolvable*.

We wish to classify unsolvable problems by measuring the "amount of unsolvability" which is inherent in them.

Definition. The *Muchnik degree* of P is the equivalence class of P under mutual Muchnik reducibility. The Muchnik degrees are partially ordered by Muchnik reducibility. They form a complete distributive lattice, denoted $\mathcal{D}_w$.

All solvable problems are of the same Muchnik degree, denoted $\mathbf{0}$. The Muchnik degree of an unsolvable problem is $> \mathbf{0}$. In general, the Muchnik degree of a problem is viewed as a measure of its *degree of difficulty* or *degree of unsolvability*.

Definition. A set of real numbers is said to be *effectively closed* if it is the complement of the union of a computable sequence of basic open sets.

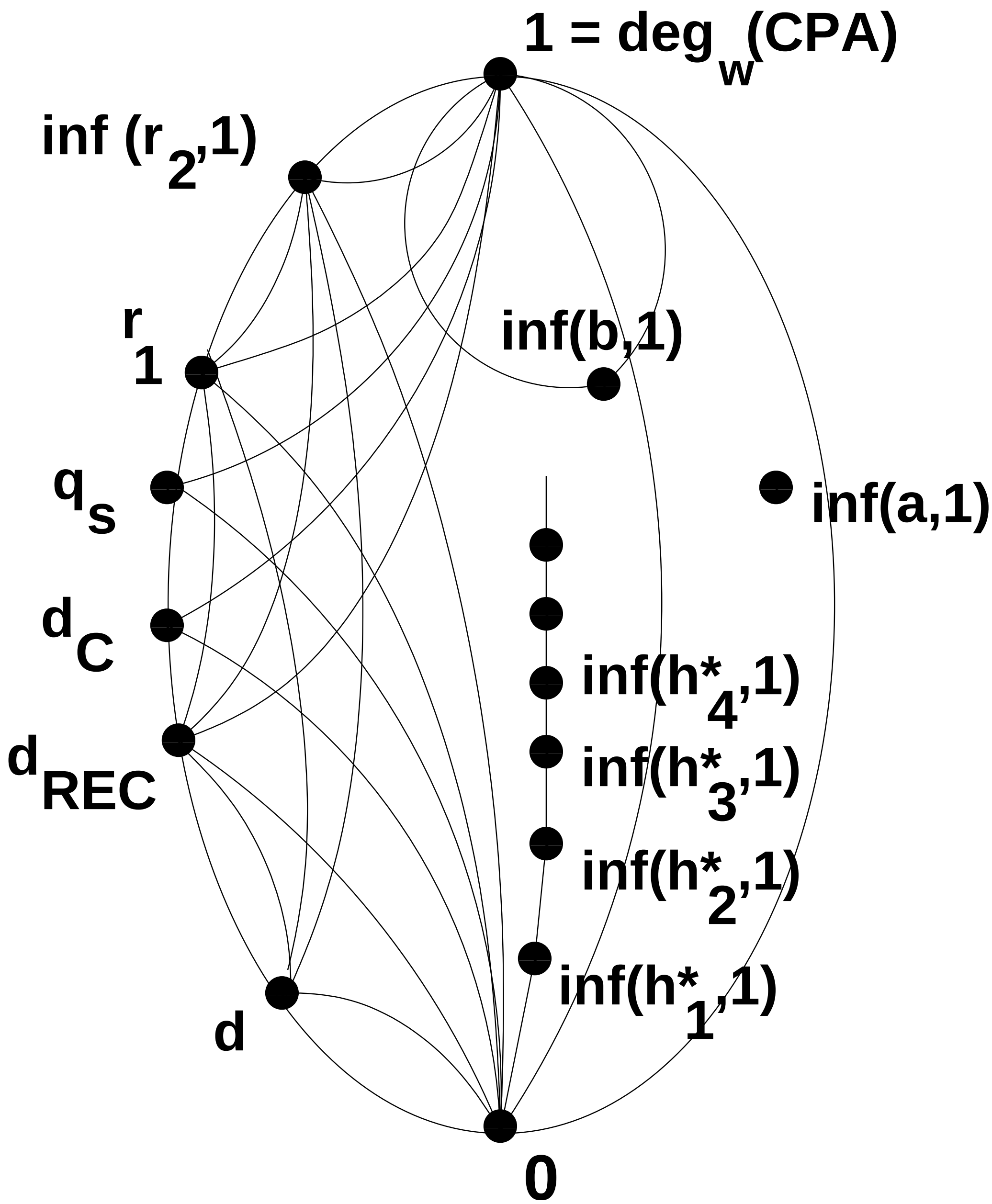
Since 1999 I have been studying the lattice of Muchnik degrees of nonempty effectively closed sets of real numbers.

This sublattice of $\mathcal{D}_w$ is denoted $\mathcal{P}_w$.

Beginning in 1999 I discovered that $\mathcal{P}_w$ contains a number of specific, natural degrees corresponding to various interesting topics in the foundations of mathematics and the foundations of computer science.

Among these topics are:

- algorithmic randomness
- the Gödel incompleteness phenomenon
- reverse mathematics
- almost everywhere domination
- diagonal nonrecursiveness
- hyperarithmeticity
- resource-bounded computational complexity
- Kolmogorov complexity
- effective Hausdorff dimension
- subrecursive hierarchies



A picture of $\mathcal{P}_w$. Here $\mathbf{a} =$ any r.e. degree, $\mathbf{h} =$ hyperarithmeticity, $\mathbf{r} =$ randomness, $\mathbf{b} =$ a.e. domination, $\mathbf{q} =$ dimension, $\mathbf{d} =$ diagonalization, $\mathbf{1} =$ incompleteness.

2-dimensional symbolic dynamics:

Let A be a finite set of symbols.

Let $A^{\mathbb{Z}\times\mathbb{Z}}$ be the set of doubly bi-infinite double sequences of symbols from A.

This is a compact metrizable space.

Points of $A^{\mathbb{Z}\times\mathbb{Z}}$ may be viewed as *tilings of the plane*, in the sense of Wang 1961.

Tiling problems were studied by logicians during the years 1960–1980.

The connection with dynamical systems was noticed only relatively recently.

The *full 2-dimensional shift* on A is the dynamical system consisting of $A^{\mathbb{Z}\times\mathbb{Z}}$ with shift operators $S_1, S_2 : A^{\mathbb{Z}\times\mathbb{Z}} \to A^{\mathbb{Z}\times\mathbb{Z}}$ given by $S_1(x)(m,n) = x(m+1,n)$ and $S_2(x)(m,n) = x(m,n+1)$.

Symbolic dynamics (continued):

A *2-dimensional subshift* on A is a nonempty closed set $X \subseteq A^{\mathbb{Z}\times\mathbb{Z}}$ which is invariant under S_1 and S_2.

Note that (X, S_1, S_2) is a 2-dimensional dynamical system. It is a subsystem of the full 2-dimensional shift on A.

Every 2-dimensional subshift X is defined by a set E of excluded configurations.

If E is finite, X is said to be *of finite type*.

Here, by a *configuration* we mean a "2-dimensional word," i.e., a member of $A^{\{1,\ldots,r\}\times\{1,\ldots,r\}}$ for some positive integer r.

2-dimensional subshifts of finite type are important in dynamical systems theory. An example is the Ising model in mathematical physics.

History:

Berger 1966 answered a question of Wang 1961 by constructing a 2-dimensional subshift of finite type with no periodic points.

Berger 1966 showed that it is undecidable whether a given finite set of excluded configurations defines a (nonempty!) 2-dimensional subshift.

Myers 1974 constructed
a 2-dimensional subshift of finite type
with no recursive points.

Hochman/Meyerovitch 2007 proved:
a real number $h \geq 0$ is the entropy of
a 2-dimensional subshift of finite type if and only if h is *right recursively enumerable*.
This means that h is the limit of a recursive decreasing sequence of rational numbers.

Muchnik degrees and symbolic dynamics:

Recently, using methods of Robinson 1971 and Myers 1974, I proved:

Theorem 1 (Simpson 2007)**.** The Muchnik degrees of 2-dimensional subshifts of finite type are the same as the Muchnik degrees of nonempty effectively closed sets of reals.

These are precisely the degrees in $\mathcal{P}_w$.

A new research program:

If X is a 2-dimensional subshift of finite type, there are surely some interesting relationships between the dynamical properties of X and the Muchnik degree of X.

These relationships remain to be explored.

Once again:

Theorem 1 (Simpson 2007)**.** The Muchnik degrees of 2-dimensional subshifts of finite type are the same as the Muchnik degrees of nonempty effectively closed sets of reals.

These are precisely the degrees in $\mathcal{P}_w$.

Theorem 1 is useful, because we can then apply known results concerning $\mathcal{P}_w$ to study 2-dimensional subshifts of finite type.

Below we present one such application.

Our application is stated purely in terms of 2-dimensional subshifts of finite type, with no mention of Muchnik degrees and no mention of recursion theory.

An application:

To state our application, we need some easy definitions which make perfect sense for all dynamical systems.

Definition. Let X and Y be 2-dimensional subshifts on k and l symbols respectively. The Cartesian product $X \times Y$ and the disjoint union $X + Y$ are 2-dimensional subshifts on kl and $k + l$ symbols respectively.

Definition. Let (X, S_1, S_2) be a 2-dimensional subshift on k symbols. Let a, b, c, d be integers with $ad - bc \neq 0$. Then, the system $(X, S_1^a S_2^b, S_1^c S_2^d)$ is canonically isomorphic to a 2-dimensional subshift on $k^{|ad-bc|}$ symbols.

Definition. If $\mathcal{U}$ is a set of 2-dimensional subshifts, let cl($\mathcal{U}$) be the closure of $\mathcal{U}$ under the above operations.

Definition. If X and Y are 2-dimensional subshifts, a *shift morphism* from X to Y is a continuous mapping $F : X \to Y$ which commutes with the shift operators.

In other words, $F(S_1(x)) = S_1(F(x))$ and $F(S_2(x)) = S_2(F(x))$ for all $x \in X$.

Now for the application.

Theorem 2 (Simpson 2007)**.**
There is an infinite set $\mathcal{W}$ of 2-dimensional subshifts of finite type, such that for any partition $\mathcal{U}, \mathcal{V}$ of $\mathcal{W}$, and for any $X \in \mathrm{cl}(\mathcal{U})$ and $Y \in \mathrm{cl}(\mathcal{V})$, there is no shift morphism from X to Y or vice versa.

Theorem 2 follows from Theorem 1 plus a known recursion-theoretic result:

> There is an infinite set of degrees in $\mathcal{P}_w$ which are lattice-theoretically independent.

This known recursion-theoretic result is proved by means of a priority argument.

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Some of my papers are available at
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Transparencies for my talks are available at
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