Bounded Limit Recursiveness

Stephen G. Simpson

Pennsylvania State University
NSF grant DMS-0600823
http://www.math.psu.edu/simpson/

simpson@math.psu.edu

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Abstract:

Let X be a Turing oracle. A function $f(n)$ is said to be boundedly limit recursive in X if it is the limit of an X-recursive sequence of X-recursive functions $\widetilde{f}(n,s)$ such that the number of times $\widetilde{f}(n,s)$ changes is bounded by a recursive function of n. Let us say that X is BLR-low if every function which is boundedly limit recursive in X is boundedly limit recursive in 0. This is a lowness property in the sense of Nies. These notions were introduced by Joshua A. Cole and the speaker in a recently submitted paper on mass problems and hyperarithmeticity. The purpose of this talk is to compare BLR-lowness to similar properties which have been considered in the recursion-theoretic literature. Among the properties discussed are: K-triviality, superlowness, jump-traceability, weak jump-traceability, total ω-recursive enumerability, array recursiveness, array jump-recursiveness, and strong jump-traceability.

Definition.

If X is a Turing oracle, let BLR(X) be the set of number-theoretic functions $f : \omega \to \omega$ which are <u>*boundedly limit recursive in* X</u>.

This means that there exist
an X-*recursive approximating function* $\widetilde{f}(n,s)$
and a *recursive bounding function* $\widehat{f}(n)$
such that

$$f(n) = \lim_s \widetilde{f}(n,s)$$

and

$$|\{s \mid \widetilde{f}(n,s) \neq \widetilde{f}(n,s+1)\}| < \widehat{f}(n)$$

for all n.

In particular, BLR(0) $= \{f \mid f \leq_{\text{wtt}} 0'\}$.

The BLR operator was introduced in

> Mass problems and hyperarithmeticity, by Joshua A. Cole and Stephen G. Simpson, 20 pages, submitted 2006 to JML.

Cole and Simpson used the BLR operator to construct a natural embedding of the hyperarithmetical hierarchy into $\mathcal{P}_{\mathsf{w}}$.

Namely, we proved that the Muchnik degrees $\inf(\mathbf{h}_\alpha^*, \mathbf{1})$ for $\alpha < \omega_1^{\mathsf{CK}}$ are distinct $\in \mathcal{P}_{\mathsf{w}}$.

Explanations:

1. For each $\alpha < \omega_1^{\mathsf{CK}}$, $\mathbf{h}_\alpha$ is the Muchnik degree of $\{0^{(\alpha)}\}$ (i.e., the hyp. hierarchy).

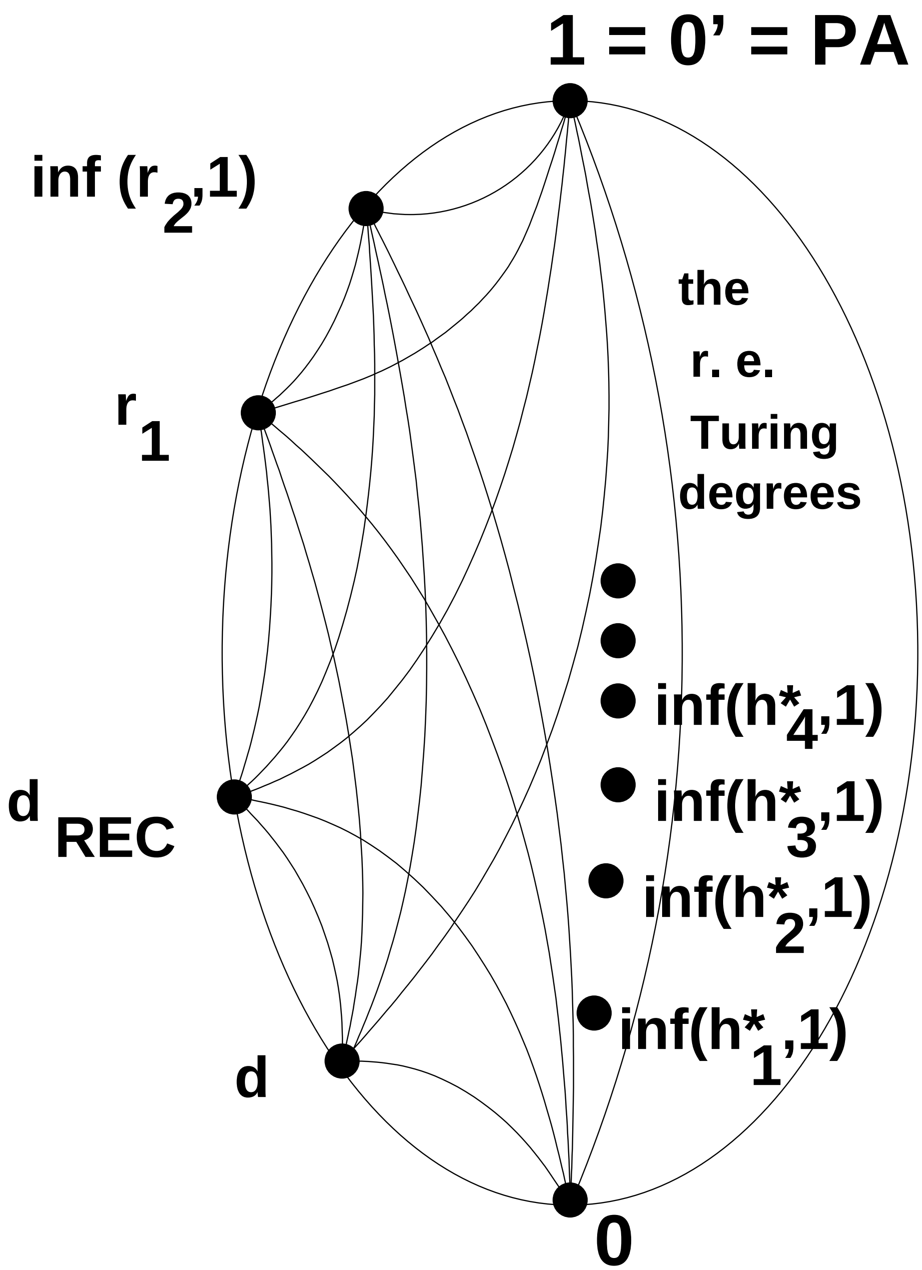
2. If $\mathbf{s} =$ the Muchnik degree of S, then $\mathbf{s}^* =$ the Muchnik degree of S^* where

   $S^* = \{Y \mid \exists X\, (X \in S \wedge \mathsf{BLR}(X) \subseteq \mathsf{BLR}(Y))\}$.

   Key Lemma: If S is Σ^0_3 then S^* is Σ^0_3.

3. $\mathcal{P}_{\mathsf{w}} =$ the lattice of Muchnik degrees of mass problems associated with nonempty Π^0_1 subsets of 2^ω.

   The top and bottom degrees in $\mathcal{P}_{\mathsf{w}}$ are denoted $\mathbf{1}$ and $\mathbf{0}$, respectively.



The Muchnik degrees inf$(\mathbf{h}^*_\alpha, \mathbf{1})$, $1 \le \alpha < \omega_1^{\mathsf{CK}}$, are incomparable with $\mathbf{d}$, $\mathbf{d}_{\mathsf{REC}}$, $\mathbf{r}_1$, inf$(\mathbf{r}_2, \mathbf{1})$, and all r.e. Turing degrees except $\mathbf{0}$ and $\mathbf{0}'$.

After Cole/Simpson, it was natural to study the star operator for its own sake. We have $\mathbf{s}^{**} = \mathbf{s}^* \leq \mathbf{s}$, $\mathsf{inf}(\mathbf{s},\mathbf{t})^* = \mathsf{inf}(\mathbf{s}^*,\mathbf{t}^*)$, $\mathcal{P}_{\mathsf{w}}^* \subseteq \mathcal{P}_{\mathsf{w}}$.

Question. For which $\mathbf{s}$ is $\mathbf{s}^* = \mathbf{0}$?

Say that X is BLR-*low* if BLR(X) $\subseteq$ BLR(0). This is a lowness property a la Nies.

For $\mathbf{s} =$ the Muchnik degree of S, we have $\mathbf{s}^* = \mathbf{0}$ if and only if $(\exists X \in S)\,(X$ is BLR-low$)$.

Theorem (Cole/Simpson). X is BLR-low iff X is superlow and jump-traceable.

Recall that *superlow* means $X' \leq_{\mathsf{tt}} 0'$, and *jump-traceable* means $(\exists$ rec fcns $p,q)\,(\forall n)$ $((\varphi_n^X(n)\downarrow \Rightarrow \varphi_n^X(n) \in W_{p(n)}) \wedge |W_{p(n)}| < q(n))$.

Nies (Advances, LC 2002) proved that neither property implies the other, though they are equivalent in the r.e. case. Moreover, every low-for-random has both properties. Thus, by Cole/Simpson, low-for-random implies BLR-low.

A weaker property than jump-traceability is *r.e.-traceability*: ($\exists$ rec fcn q) $(\forall f \leq_{\mathsf{T}} X)$ ($\exists$ rec fcn p) $\forall n\,(f(n) \in W_{p(n)} \wedge |W_{p(n)}| < q(n))$.

The following theorem is due to Kjos-Hanssen/Merkle/Stephan, STACS 2006.

Theorem. If X is r.e.-traceable, then X is DNR-*free*, i.e., there is no diagonally nonrecursive function $\leq_{\mathsf{T}} X$.

(The proof is short but ingenious.)

Thus, letting $\mathbf{d} =$ the Muchnik degree of the mass problem associated with diagonal nonrecursiveness, we have $\mathbf{d}^* > \mathbf{0}$.

It follows that $\mathbf{s}^* > \mathbf{0}$ for all $\mathbf{s} \geq \mathbf{d}$.

On the other hand,
by the results of Nies and Cole/Simpson,
we can find $\mathbf{s}, \mathbf{t} \in \mathcal{P}_{\mathsf{w}}$ such that
$\sup(\mathbf{s}, \mathbf{t}) = \mathbf{1}$, hence $\sup(\mathbf{s}, \mathbf{t})^* = \mathbf{1}^* > \mathbf{0}$,
yet $\mathbf{s}^* = \mathbf{t}^* = \mathbf{0}$.

One thing leads to another.

A weaker property than r.e.-traceability is *weak r.e.-traceability*:

$(\forall f \leq_T X)$ ($\exists$ recursive functions p, q) $(\forall n)\,(f(n) \in W_{p(n)} \wedge |W_{p(n)}| < q(n))$.

Obviously, if X is hyperimmune-free, then X is weakly r.e.-traceable. Moreover, by the Hyperimmune-Free Basis Theorem, there exists a hyperimmune-free X which is of PA degree, hence not DNR-free.

Thus, the Kjos-Hanssen/Merkle/Stephan result does not hold with "r.e.-traceable" replaced by "weakly r.e.-traceable".

However, we have:

Theorem (Kjos-Hanssen, unpublished).

If X is weakly r.e.-traceable and of hyperimmune degree, then X is DNR-free.

Theorem (Kjos-Hanssen, unpublished).

If X is weakly r.e.-traceable and of hyperimmune degree, then X is DNR-free.

Proof.

Suppose X is not DNR-free.

By an argument of Jockusch 1989, for all $g \leq_T X$ we can find $h \leq_T X$ such that $h(n) \not\simeq \varphi_i(i)$ for all $i < g(n)$.

Since X is of hyperimmune degree, let $g \leq_T X$ be recursively unbounded.

Since X is weakly r.e.-traceable, let p and q be recursive functions such that $h(n) \in W_{p(n)}$ and $|W_{p(n)}| < q(n)$ for all n.

Let $s(j,n)$ be a recursive function such that $\varphi_{s(j,n)}(s(j,n)) \simeq$ the jth member of $W_{p(n)}$ in order of recursive enumeration.

Since g is recursively unbounded, let n be such that $g(n) > \max\{s(j,n) \mid j < q(n)\}$.

Since $|W_{p(n)}| < q(n)$, it follows that $h(n) \notin W_{p(n)}$, a contradiction, Q.E.D.

We write $\text{REC}(X) = \{f \mid f \leq_{\text{T}} X\}$.

Corollary (Kjos-Hanssen, unpublished).

If $\text{REC}(X) \subseteq \text{BLR}(0)$, then X is DNR-free.

Proof. The hypothesis easily implies that X is weakly r.e.-traceable. Moreover, since $X \leq_{\text{T}} 0'$, X is either recursive or of hyperimmune degree. If follows by the previous theorem that X is DNR-free, Q.E.D.

The following paradox was conjectured by Simpson and proved by Barmpalias.

Corollary (Barmpalias, unpublished).

$\exists f : \omega \to \omega$ such that $f' \leq_{\text{wtt}} 0'$ yet $f \not\leq_{\text{wtt}} 0'$.

Proof (Kjos-Hanssen, unpublished).

By the Superlow Basis Theorem,
let X be superlow of PA degree.
Since X is not DNR-free, apply the previous corollary to get $f \leq_{\text{T}} X$ with $f \notin \text{BLR}(0)$.
Then $f' \leq_{\text{m}} X' \leq_{\text{tt}} 0'$, hence $f' \leq_{\text{wtt}} 0'$,
but $f \not\leq_{\text{wtt}} 0'$, Q.E.D.

Recall from Downey/Jockusch/Stob 1996 the property of *array recursiveness*:

$(\exists g \leq_{\mathrm{wtt}} 0')\,(\forall f \leq_{\mathrm{T}} X)\,(g$ dominates $f)$.

We consider two variant properties:

1. *weak array recursiveness*:

$(\forall f \leq_{\mathrm{T}} X)\,(\exists g \leq_{\mathrm{wtt}} 0')\,(g$ dominates $f)$.

2. *strong array recursiveness*:

$(\exists g \leq_{\mathrm{wtt}} 0')\,(\forall n)\,(\varphi_n^X(n)\downarrow \Rightarrow \varphi_n^X(n) < g(n))$.

Theorem (Simpson, unpublished). If X is r.e., then X is weakly array recursive iff $\mathrm{REC}(X) \subseteq \mathrm{BLR}(0)$.

Remark. Downey/Greenberg/Weber 2006 have shown that this class of r.e. degrees is naturally lattice-theoretically definable in the r.e. degrees, as those which do not bound a critical triple.

Remark. If X is not r.e., the theorem can fail badly. See the first corollary below.

First strengthen the Superlow Basis Theorem.

Theorem (Simpson, unpublished). If $P \subseteq 2^\omega$ is nonempty Π^0_1, there exists $X \in P$ which is superlow and strongly array recursive.

Proof.

Write $P \upharpoonright F = \{Z \in P \mid \varphi_n^Z(n) \uparrow \forall n \in F\}$.

Inductively define

$$f(n) = \begin{cases} 1 & \text{if } P \upharpoonright \{m < n \mid f(m) = 0\} \cup \{n\} = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let $X \in P \upharpoonright \{n \mid f(n) = 0\}$. Note that f is the characteristic function of $X' = \{n \mid \varphi_n^X(n) \downarrow\}$.

X is superlow, because $f(n)$ depends only on whether $P \upharpoonright F = \emptyset$ for each $F \subseteq \{0, 1, \ldots, n\}$.

X is strongly array recursive, because
given $n \in X'$ and $F_n = \{m < n \mid f(m) = 0\}$,
we have $\varphi_n^Z(n) \downarrow$ for all $Z \in P \upharpoonright F_n$,
hence we can compute
$g(n) =$ an upper bound for these values.

Q.E.D.

Corollary (Simpson, unpublished).

There exists X which is superlow and strongly array recursive, yet REC(X) $\not\subseteq$ BLR(0).

Proof. By our basis theorem above, let X be superlow, strongly array recursive, and of PA degree. Since X is not DNR-free, it follows by Kjos-Hanssen that REC(X) $\not\subseteq$ BLR(0).

The same example shows:

Corollary (Simpson, unpublished).

There exists X which is superlow and strongly array recursive, yet not weakly r.e.-traceable.

Remark. Much more can be said about lowness and tameness properties of arbitrary Turing oracles (not necessarily r.e.). See also the talks at this meeting by George Barmpalias, Noam Greenberg, and Denis Hirschfeldt.

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