Reverse Mathematics and its Significance for Mathematical Logic

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Outline of this talk:

1. Foundations of mathematics (= f.o.m.).

2. Mathematical logic.

3. The FOM mailing list.

5. History of reverse mathematics (= r.m.).

4. Themes of reverse mathematics.

6. Foundational aspects of r.m.

7. Using r.m. to enrich mathematical logic.

8. Using r.m. to enrich recursion theory.

9. An extension of the r.e. Turing degrees: Muchnik degrees of Π^0_1 subsets of 2^ω.

10. A symmetric ω-model of WKL_0.

11. A symmetric β-model of ATR_0.

Foundations of mathematics (f.o.m.):

Foundations of mathematics is the study of the most basic concepts and logical structure of mathematics as a whole.

Among the most basic mathematical concepts are:

number, set, function, algorithm, mathematical definition, mathematical proof, mathematical theorem, mathematical axiom.

Some big names in f.o.m. are:

Aristotle, Euclid, Descartes, Leibniz, . . . , Dedekind, Cantor, Frege, Russell, Zermelo, Hilbert, Weyl, Brouwer, Skolem, Gödel, Church, Turing, Post, Kleene, . . .

A key f.o.m. question:

What are the appropriate axioms for mathematics?

Part A of Friedman's talk pushed the envelope, showing that unusually strong set-theoretic axioms can be useful in the development of ordinary mathematics.

Reverse Mathematics examines a different variant of the same question:

What axioms are needed to prove specific, known theorems of ordinary mathematics?

The Gödel Hierarchy:

$$
\begin{array}{rl}
\text{strong} & \left\{\begin{array}{l}
\vdots \\
\text{supercompact cardinal} \\
\vdots \\
\text{measurable cardinal} \\
\vdots \\
\text{ZFC (ZF set theory with choice)} \\
\text{Zermelo set theory} \\
\text{simple type theory}
\end{array}\right. \\[2em]
\text{medium} & \left\{\begin{array}{l}
\mathsf{Z}_2 \text{ (2nd order arithmetic)} \\
\vdots \\
\Pi^1_2 \text{ comprehension} \\
\Pi^1_1 \text{ comprehension} \\
\mathsf{ATR}_0 \text{ (arith. transfinite recursion)} \\
\mathsf{ACA}_0 \text{ (arithmetical comprehension)}
\end{array}\right. \\[2em]
\text{weak} & \left\{\begin{array}{l}
\mathsf{WKL}_0 \text{ (weak König's lemma)} \\
\mathsf{RCA}_0 \text{ (recursive comprehension)} \\
\text{PRA (primitive recursive arithmetic)} \\
\text{EFA (elementary arithmetic)} \\
\text{bounded arithmetic} \\
\vdots
\end{array}\right.
\end{array}
$$

Mathematical logic:

In the post-war period, f.o.m. mutated into a different subject, *mathematical logic*, which largely lost touch with its f.o.m. roots.

The 4 main subdivisions of mathematical logic are:

1. Model theory.

2. Set theory.

3. Recursion theory.

4. Proof theory.

For an overview, see *Handbook of Mathematical Logic*, edited by J. Barwise, 1977, XI + 1165 pages.

Each of the 4 has become isolated from the others. E.g., the panels on logic in the 20th and 21st centuries, at ASL 2000 in Urbana.

The FOM mailing list:

FOM is an automated e-mail list for discussing foundations of mathematics. There are currently more than 600 subscribers. There have been more than 5500 postings.

FOM is maintained and moderated by S. Simpson. The FOM Editorial Board consists of M. Davis, H. Friedman, C. Jockusch, D. Marker, S. Simpson, A. Urquhart.

FOM postings and information are available on the web at

`http://www.math.psu.edu/simpson/fom/`

Friedman and Simpson founded FOM in 1997, to promote a controversial idea: **mathematical logic is or ought to be driven by f.o.m. considerations.**

f.o.m. = foundations of mathematics.

History of reverse mathematics:

Kreisel 1960's introduces several subsystems of 2nd order arithmetic, including Δ^1_1-CA, Σ^1_1-AC, Σ^1_1-DC, Π^1_∞-TI$_0$.

Friedman 1967 (Ph.D. thesis, MIT) introduces a system equivalent to ATR, to show that Σ^1_1-AC $\neq$ Σ^1_1-DC.

Simpson 1973 (Berkeley) lectures on subsystems of 2nd order arithmetic and their role in f.o.m. Printed lecture notes.

Steel 1973 shows that ATR $\leftrightarrow$ comparability of countable well orderings, over Δ^1_1-CA. This and other r.m. results appear in Steel's Ph.D. thesis, supervised by Simpson.

Friedman 1974 (ICM lecture) states the first theme of reverse mathematics. Friedman 1975 (two JSL abstracts) introduces systems with restricted induction, including RCA$_0$.

History of r.m. (continued):

Simpson 1982 (Cornell recursion theory meeting) highlights the "big five" of r.m.: RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0.

Key r.m. papers by Simpson 1984 and Friedman/Simpson/Smith 1985 on reverse analysis, reverse algebra respectively.

Simpson from 1977 onward supervises numerous Ph.D. theses at Penn State: Smith, Brackin, Ferreira, Hirst, Brown, Yu, Marcone, Humphreys, Giusto, . . . ,
and publishes numerous papers . . . .

Simpson 1998 finishes his book on subsystems of 2nd order arithmetic and reverse mathematics.

Simpson 1999 begins assembling the companion volume,
Reverse Mathematics 2001.

Logicians who have contributed to r.m.:

Jockusch, Shore, Lempp, Cholak, Slaman (also in his Friedman persona), Downey, Cenzer, Remmel, Solomon, Hirschfeldt, Sieg, Avigad, Kohlenbach, Schmerl, Kossak, Steel, Becker, Velleman, Smith, Brackin, Brown, Yu, Hirst, Marcone, Humphreys, Giusto, Tanaka, Yamazaki, Ferreira, Fernandes, Mourad, Chong, Yang, . . .

Two books on reverse mathematics:

1.

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Perspectives in Mathematical Logic
Springer-Verlag, 1999
XIV + 445 pages

`http://www.math.psu.edu/simpson/sosoa/`

2.

S. G. Simpson (editor)
Reverse Mathematics 2001

A volume of papers by various authors,
to appear,
approximately 400 pages.

`http://www.math.psu.edu/simpson/revmath/`

Current framework of reverse mathematics:

Second order arithmetic ($= \mathsf{Z}_2$) is a two-sorted system.

Number variables $m, n, \ldots$ range over

$$\omega = \{0, 1, 2, \ldots\}\ .$$

Set variables $X, Y, \ldots$ range over subsets of ω.

We have $+$, $\times$, $=$ on ω, plus the membership relation

$$\in \quad = \quad \{(n, X) : n \in X\} \quad \subseteq \quad \omega \times P(\omega)\ .$$

Within subsystems of Z_2, we can formalize contemporary rigorous mathematics (analysis, algebra, geometry, combinatorics, . . .).

Subsystems of Z_2 are the basis of our current understanding of the logical structure of contemporary mathematics.

Themes of reverse mathematics:

Let τ be a mathematical theorem. Let S_τ be the weakest natural subsystem of Z_2 in which τ is provable.

1. Very often, the principal axiom of S_τ is logically equivalent to τ.

2. Furthermore, only a few subsystems of Z_2 arise in this way.

For a full exposition, see my book *Subsystems of Second Order Arithmetic*, Springer, 1999, XIV + 445 pages.

Themes of r.m. (continued):

We develop a table indicating which mathematical theorems can be proved in which subsystems of Z_2.

	RCA_0	WKL_0	ACA_0	ATR_0	Π^1_1-CA_0
analysis (separable):					
differential equations	X	X			
continuous functions	X, X	X, X	X		
completeness, etc.	X	X	X		
Banach spaces	X	X, X			X
open and closed sets	X	X		X, X	X
Borel and analytic sets	X			X, X	X, X
algebra (countable):					
countable fields	X	X, X	X		
commutative rings	X	X	X		
vector spaces	X		X		
Abelian groups	X		X	X	X
miscellaneous:					
mathematical logic	X	X			
countable ordinals	X		X	X, X	
infinite matchings		X	X	X	
the Ramsey property			X	X	X
infinite games			X	X	X

The Big 5 in the Gödel Hierarchy:

strong:

$\vdots$

supercompact cardinal

$\vdots$

measurable cardinal

$\vdots$

ZFC (ZF set theory with choice)

Zermelo set theory

simple type theory

medium:

Z_2 (2nd order arithmetic)

$\vdots$

Π^1_2 comprehension

- Π^1_1 comprehension
- ATR_0 (arith. transfinite recursion)
- ACA_0 (arithmetical comprehension)

weak:

- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)

PRA (primitive recursive arithmetic)

EFA (elementary arithmetic)

bounded arithmetic

$\vdots$

Foundational consequences of r.m.:

1. We precisely classify mathematical theorems, according to which subsystems of Z_2 they are provable in.

2. We identify certain subsystems of Z_2 as being mathematically natural. The naturalness is rigorously demonstrated.

3. We work out the consequences of particular foundational doctrines:

   - recursive analysis (Pour-El/Richards)
   - constructivism (Bishop)
   - finitistic reductionism (Hilbert)
   - predicativism (Weyl)
   - predicative reductionism (Feferman/Friedman/Simpson)
   - impredicative analysis (Takeuti/Schütte/Pohlers)

Foundational consequences (continued):

By means of reverse mathematics, we identify five particular subsystems of Z_2 as being mathematically natural. We correlate these systems to traditional f.o.m. programs.

RCA_0	constructivism	Bishop
WKL_0	finitistic reductionism	Hilbert
ACA_0	predicativism	Weyl, Feferman
ATR_0	predicative reductionism	Friedman, Simpson
Π^1_1-CA_0	impredicativity	Feferman et al.

We analyze these foundational proposals in terms of their consequences for mathematical practice. Under the various proposals, which mathematical theorems are "lost"?

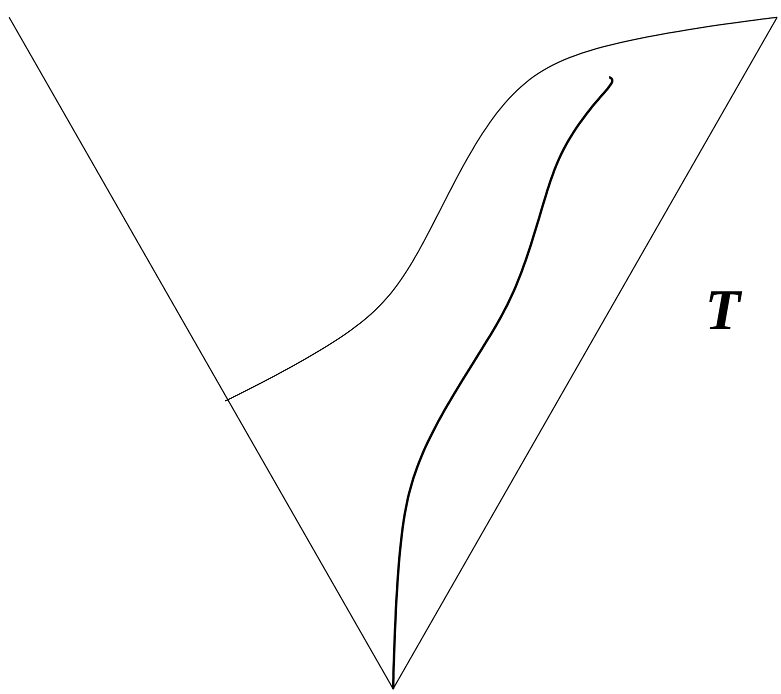
Reverse mathematics provides precise answers to such questions.

$\mathsf{RCA}_0 = \Sigma^0_1$ induction
$+\, \Delta^0_1$ (i.e., recursive) comprehension
("formalized recursive mathematics")

$\mathsf{WKL}_0 = \mathsf{RCA}_0 +$ Weak König's Lemma:

Every infinite subtree of the full binary tree of finite sequences of 0's and 1's has an infinite path.

(a "formalized compactness principle")



Reverse Mathematics for WKL_0**:**

WKL_0 is equivalent over RCA_0 to each of the following mathematical statements:

1. The Heine/Borel Covering Lemma: Every covering of $[0,1]$ by a sequence of open intervals has a finite subcovering.

2. Every covering of a compact metric space by a sequence of open sets has a finite subcovering.

3. Every continuous real-valued function on $[0,1]$ (or on any compact metric space) is bounded (uniformly continuous, Riemann integrable).

6. The Maximum Principle: Every continuous real-valued function on $[0,1]$ (or on any compact metric space) has (or attains) a supremum.

R. M. for WKL_0 (continued):

7. The local existence theorem for solutions of (finite systems of) ordinary differential equations.

8. Gödel's Completeness Theorem: every finite (or countable) set of sentences in the predicate calculus has a countable model.

9. Every countable commutative ring has a prime ideal.

10. Every countable field (of characteristic 0) has a unique algebraic closure.

11. Every countable formally real field is orderable.

12. Every countable formally real field has a (unique) real closure.

R. M. for WKL_0 (continued):

13. Brouwer's Fixed Point Theorem: Every (uniformly) continuous function $\phi : [0,1]^n \to [0,1]^n$ has a fixed point.

14. The Separable Hahn/Banach Theorem: If f is a bounded linear functional on a subspace of a separable Banach space, and if $\|f\| \leq 1$, then f has an extension $\widetilde{f}$ to the whole space such that $\|\widetilde{f}\| \leq 1$.

15. Banach's Theorem: In a separable Banach space, given two disjoint convex open sets A and B, there exists a closed hyperplane H such that A is on one side of H and B is on the other.

16. Every countable k-regular bipartite graph has a perfect matching.

Significance of r.m.
for mathematical logic:

1. **recursion theory**
   - reversals use coding arguments
   - ω-models of subsystems of Z_2
   - Turing ideals (RCA_0)
   - Turing jump ideals (ACA_0)
   - Π^0_1 subsets of 2^ω (WKL_0)
   - hyperarithmetic theory (ATR_0, Π^1_1-CA_0)
   - basis and anti-basis theorems
   - Medvedev and Muchnik degrees

2. **model theory**
   - models of subsystems of Z_2
   - ω-models and non-ω-models
   - nonstandard models of arithmetic
   - model-theoretic conservation results

Significance of r.m. for m.l., continued:

3. **set theory**
   - r.m. highlights the foundational significance of set theory
   - ZFC is usually too strong for math, but r.m. highlights the rare cases when strong axioms are actually needed
   - set-theoretic methods are used to build models of subsystems of Z_2 (constructible sets, forcing, etc.)

4. **proof theory**
   - subsystems of Z_2 studied in proof theory
   - r.m. brings out their f.o.m. significance
   - Gentzen-style ordinal analysis
   - proof-theoretic conservation results

Outline of the rest of the talk:

Using r.m. to enrich recursion theory.

1. An intensively studied recursion-theoretic structure is the semilattice of recursively enumerable Turing degrees, $\mathcal{R}_T$. Regrettably, the study of $\mathcal{R}_T$ has been sterile for f.o.m.

We show how to enlarge and improve $\mathcal{R}_T$ in a way that provides connections to f.o.m., via reverse mathematics involving subsystems of WKL_0.

2. We use recursion theory to construct models of WKL_0 and ATR_0 that are of foundational interest.

Background: the r.e. Turing degrees

For $X, Y \subseteq \omega = \{0, 1, 2, \ldots\}$,
X is *Turing reducible* to Y (i.e., $X \leq_T Y$) iff X is computable using an oracle for Y.

The *Turing degrees* are the equivalence classes under $\leq_T$, ordered by $\leq_T$.

The l.u.b. of two Turing degrees is given by
$X \oplus Y = \{2n : n \in X\} \cup \{2n + 1 : n \in Y\}$.

$X \subseteq \omega$ is *r.e.* (i.e., *recursively enumerable*) iff it is the range of a recursive function.

An *r.e. Turing degree* is a Turing degree that contains an r.e. set.

$\mathcal{R}_T$ is the semilattice of r.e. Turing degrees. This structure has been studied extensively by recursion theorists.

Background, continued:

$\mathcal{R}_T$ is the semilattice of r.e. Turing degrees.

Intensive study of lattice-theoretic properties of $\mathcal{R}_T$ has yielded nothing for f.o.m.

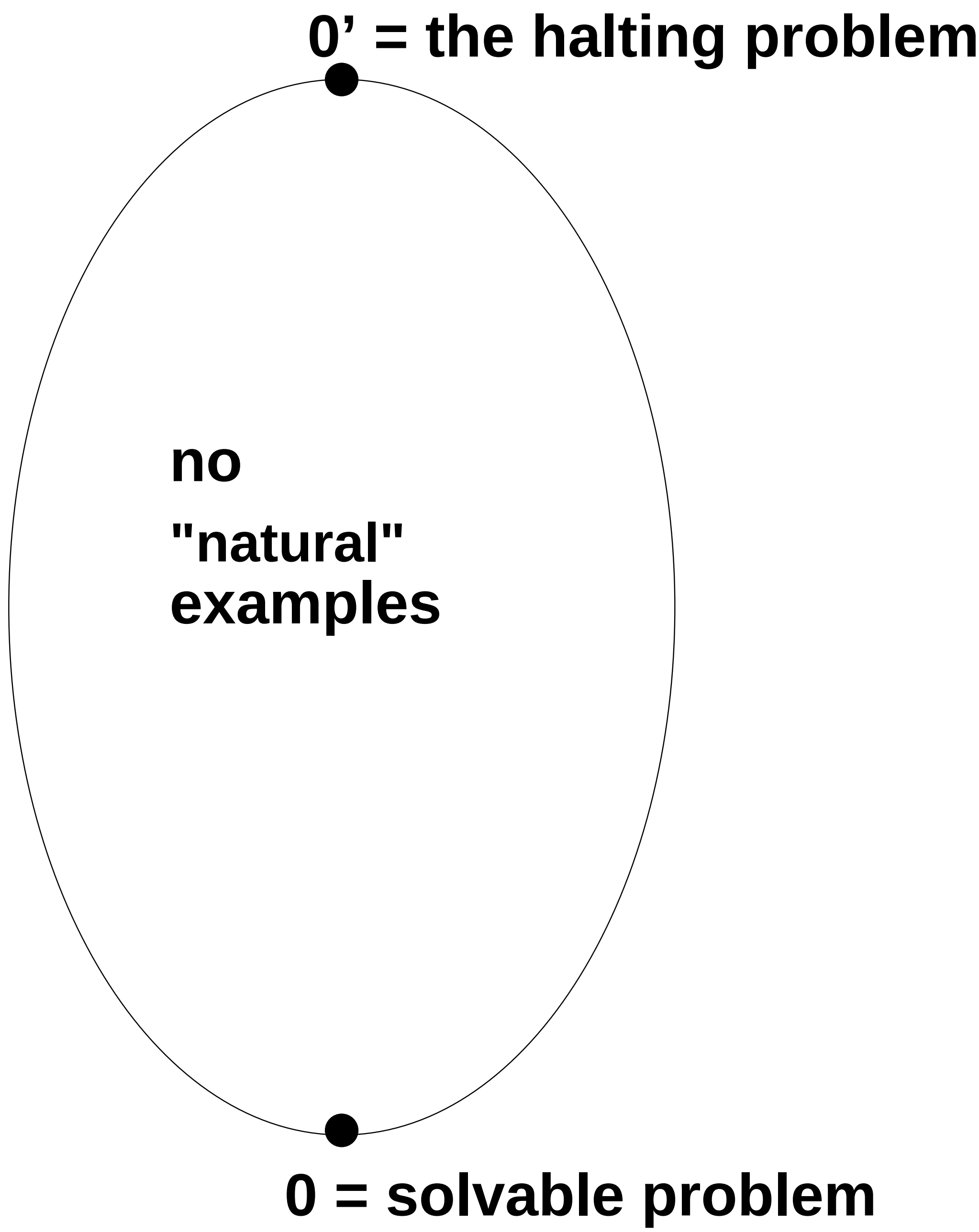
Moreover, after 50 years, the only known specific examples of r.e. Turing degrees are the bottom and top elements of $\mathcal{R}_T$.

$0 =$ Turing degree of recursive sets.

$0' =$ Turing degree of the Halting Problem.

There are infinitely many r.e. Turing degrees, but there are no known "natural" ones, other than 0 and $0'$.

A picture of the r.e. Turing degrees, $\mathcal{R}_T$:



An extension of the r.e. Turing degrees:

We define the Muchnik lattice $\mathcal{P}_w$.

The Cantor space is $2^\omega = \{X : \omega \to \{0,1\}\}$.

For $P, Q \subseteq 2^\omega$, P is *Muchnik reducible* to Q ($P \leq_w Q$) iff every member of Q computes a member of P, i.e., $(\forall Y \in Q)\,(\exists X \in P)\, X \leq_T Y$.

Muchnik degrees are equivalence classes of subsets of 2^ω under $\leq_w$, ordered by $\leq_w$.

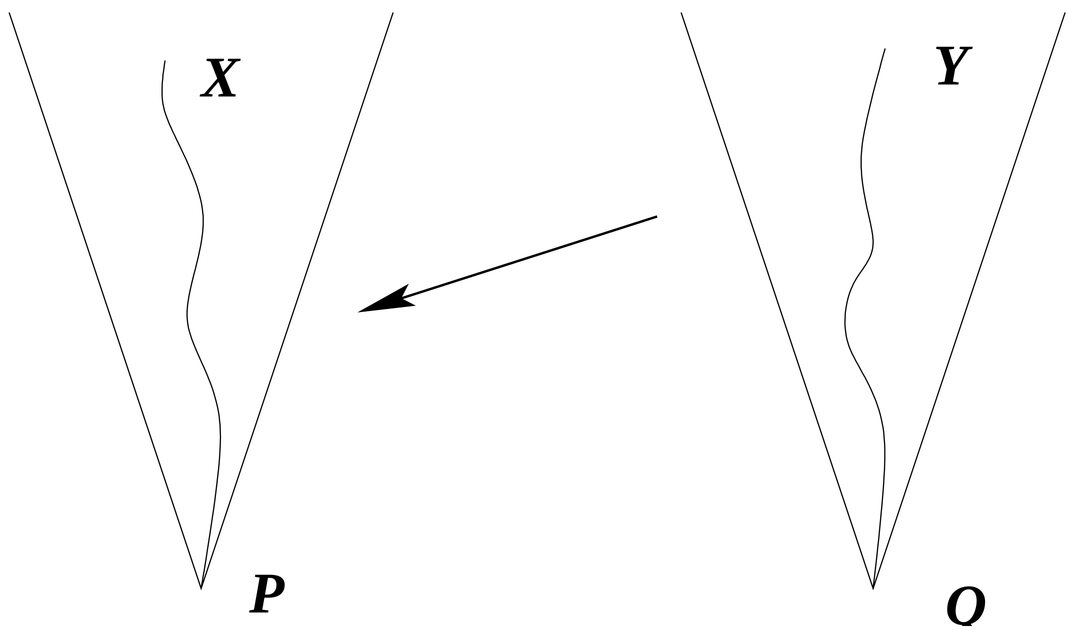
The l.u.b. of two Muchnik degrees is given by $P \times Q = \{X \oplus Y : X \in P \text{ and } Y \in Q\}$.
The g.l.b. is given by $P \cup Q$.

$P \subseteq 2^\omega$ is Π^0_1 iff P is the set of paths through a recursive subtree of the full binary tree of finite sequences of 0's and 1's.

$\mathcal{P}_w$ is the lattice of Muchnik degrees of nonempty Π^0_1 subsets of 2^ω.

(It is important here that 2^ω is compact.)

Muchnik reducibility:



$P \leq_w Q$ means:

$\forall Y \in Q \;\; \exists X \in P \;\; X \leq_T Y$.

P, Q are given by recursive subtrees of the full binary tree of finite sequences of 0's and 1's.

X, Y are infinite (nonrecursive) paths through P, Q respectively.

An extension, continued:

Properties of $\mathcal{P}_w$, the lattice of Muchnik degrees of nonempty Π^0_1 subsets of 2^ω:

First, it is a distributive lattice. Thus, its structure is more regular than that of $\mathcal{R}_T$.

It has a bottom and a top element:

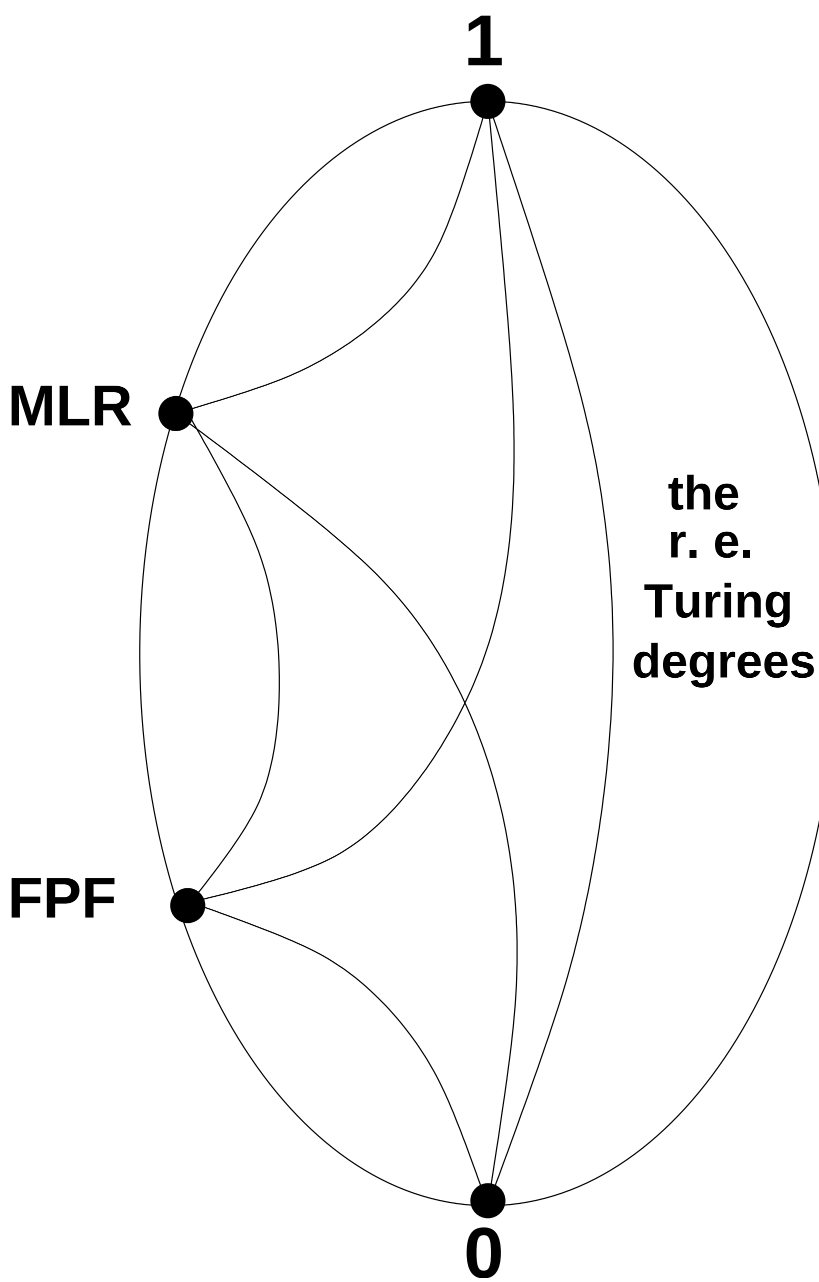
$0 =$ the Muchnik degree of 2^ω,

$1 =$ the Muchnik degree of the set of completions of theories that are *sufficiently strong*, in the sense of the Gödel/Rosser Theorem: EFA, PA, Z_2, ZFC, ...

Finally, there are at least two other "natural" Muchnik degrees in $\mathcal{P}_w$. See below.

In these important senses, the Muchnik lattice $\mathcal{P}_w$ is much better than $\mathcal{R}_T$, the semilattice of r.e. Turing degrees. It overcomes some of the well known deficiencies of $\mathcal{R}_T$.
(Simpson, August 1999, on FOM)

A picture of the Muchnik lattice $\mathcal{P}_w$:



An extension, continued:

Two “natural” Muchnik degrees in $\mathcal{P}_w$.

MLR = the Muchnik degree of the set of Martin-Löf random sequences of 0’s and 1’s. (essentially due to Kučera 1985)

FPF = the Muchnik degree of the set of fixed-point-free functions, in the sense of the Arslanov Completeness Criterion. (Simpson 2002)

We have $0 < \text{FPF} < \text{MLR} < 1$.

F.o.m. connection: The Muchnik degrees MLR and FPF correspond to subsystems of WKL_0 which arise in the Reverse Mathematics of measure theory (Yu/Simpson 1990) and continuous functions (Giusto/Simpson 2000), respectively. The Muchnik degree 1 corresponds to WKL_0 itself.

Problem: Find additional “natural” Muchnik degrees in $\mathcal{P}_w$.

MLR = the Muchnik degree of the set of Martin-Löf random reals

= the maximum Muchnik degree of a Π^0_1 subset of 2^ω of positive measure.

(implicit in Kučera 1985)

FPF = the Muchnik degree of the set of fixed-point-free functions

= the Muchnik degree of the set of diagonally non-recursive functions

= the Muchnik degree of the set of effectively immune sets

= the Muchnik degree of the set of effectively biimmune sets

(implicit in Jockusch 1989)

An extension, continued:

Further properties of the Muchnik lattice $\mathcal{P}_w$.

1. $\mathcal{P}_w$ is a countable distributive lattice. Every countable distributive lattice is lattice embeddable in every initial segment of $\mathcal{P}_w$. (Binns/Simpson 2001)

2. For all $P > 0$ there exist $P_1, P_2 < P$ such that $P =$ l.u.b.(P_1, P_2). (Stephen Binns, 2002)

3. There does not exist $P < 1$ such that l.u.b.$(P, \mathrm{MLR}) = 1$. (Simpson 2001)

4. There do not exist $P_1, P_2 > \mathrm{MLR}$ such that g.l.b.$(P_1, P_2) = \mathrm{MLR}$. (Simpson 2001)

5. If $P > 0$ is thin, then P is Muchnik incomparable with MLR. (Simpson 2001)

6. There do not exist $P_1, P_2 > 0$ such that g.l.b.$(P_1, P_2) = 0$. (trivial)

7. If $S \subseteq 2^\omega$ is Σ^0_3, then for all Π^0_1 $P \subseteq 2^\omega$ there exists Π^0_1 $Q \subseteq 2^\omega$ such that Q is Muchnik equivalent to $S \cup P$. (Simpson 2002)

8. Given $P \geq_w \mathsf{FPF}$, we have a semilattice embedding of the r.e. Turing degrees into $\mathcal{P}_w$, given by $X \mapsto \{X\} \cup P$. (Simpson 2002)

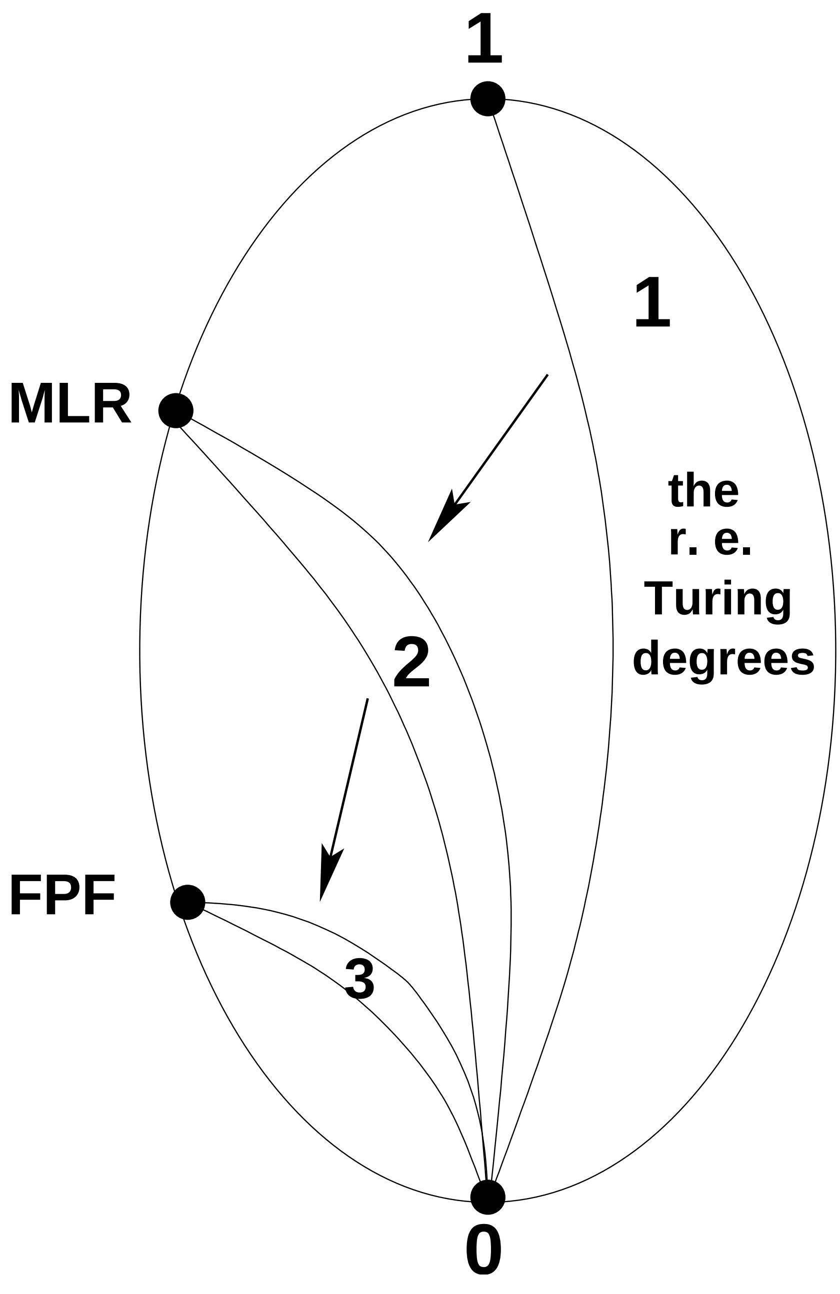
In particular, we have these three "natural" embeddings:

The *Gödel/Rosser embedding*, $X \mapsto \{X\} \cup 1$.

The *Martin-Löf embedding*, $X \mapsto \{X\} \cup \mathsf{MLR}$.

The *Arslanov embedding*, $X \mapsto \{X\} \cup \mathsf{FPF}$.

Three "natural" embeddings of the r.e. Turing degrees into the Muchnik lattice $\mathcal{P}_w$:



An extension, continued:

As we have seen,
the r.e. Turing degrees are embedded in $\mathcal{P}_w$.

Technical Note: Using a generalized Arslanov criterion, we can embed a wider class of Turing degrees: those that are $\leq 0'$ and n-REA for some $n \in \omega$.

Summary: The intensively studied semilattice of r.e. Turing degrees, $\mathcal{R}_T$ is included in the mathematically more natural, but less studied, Muchnik lattice, $\mathcal{P}_w$.

Moral: By studying the Muchnik lattice instead of the r.e. Turing degrees, recursion theorists could connect better to f.o.m.

An interesting ω-model of WKL_0:

Let $\mathcal{P}$ be the nonempty Π^0_1 subsets of 2^ω, ordered by inclusion. Forcing with $\mathcal{P}$ is known as Jockusch/Soare forcing.

Lemma (Simpson 2000). Let X be J/S generic. Suppose $Y \leq_T X$. Then (i) Y is J/S generic, and (ii) X is J/S generic relative to Y.

Theorem (Simpson 2000). There is an ω-model M of WKL_0 with the following property: For all $X, Y \in M$, X is definable from Y in M if and only if X is Turing reducible to Y.

Proof. M is obtained by iterated J/S forcing. We have

$$M \;=\; \mathsf{REC}[X_1, X_2, \ldots, X_n, \ldots]$$

where, for all n, X_{n+1} is J/S generic over $\mathsf{REC}[X_1, \ldots, X_n]$. To show that M has the desired property, we use symmetry arguments based on the Recursion Theorem.

Foundational significance of M:

The above ω-model, M, represents a compromise between the conflicting needs of

(a) recursive mathematics ("everything is computable")

and

(b) classical rigorous mathematics as developed in WKL_0 ("every continuous real-valued function on $[0,1]$ attains a maximum", "every countable commutative ring has a prime ideal", etc etc).

Namely, M contains enough nonrecursive objects for WKL_0 to hold, yet the recursive objects form the "definable core" of M.

Foundational significance (continued):

More generally, consider the scheme

> (*) For all X and Y, if X is definable from Y then X is recursive in Y.

in the language of 2nd order arithmetic.

Often in mathematics, under some assumptions on a given countably coded object X, there exists a unique countably coded object Y having some property stated in terms of X. In this situation, (*) implies that Y is Turing computable from X. This is of obvious f.o.m. significance.

Simpson 2000 shows that, for every countable model of WKL_0, there exists a countable model of $\mathsf{WKL}_0 + (*)$ with the same first order part. Thus $\mathsf{WKL}_0 + (*)$ is conservative over WKL_0 for first order arithmetical sentences.

Hyperarithmetical analogs:

Theorem (Simpson 2000). There is a countable β-model M such that, for all $X, Y \in M$, X is definable from Y in M if and only if X is hyperarithmetical in Y.

In the language of second order arithmetic, consider the scheme

> (**) for all X, Y, if X is definable from Y, then X is hyperarithmetical in Y.

Theorem (Simpson 2000).

1. $\mathsf{ATR}_0 + (**)$ is conservative over ATR_0 for Σ^1_2 sentences.
2. Π^1_∞-$\mathsf{TI}_0 + (**)$ is conservative over Π^1_∞-TI_0 for Σ^1_2 sentences.

Some of my papers are available at
`http://www.math.psu.edu/simpson/papers/`.

Transparencies for my talks are available at
`http://www.math.psu.edu/simpson/talks/`.

THE END