Why the Recursion Theorists Ought to Thank Me

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Outline of this talk:

1. Foundations of mathematics (= f.o.m.).
2. The FOM mailing list.
3. History of recursion theory (= r.t.), a.k.a. computability theory.
4. Two leading recursion theorists: Sacks and Soare.
5. History of reverse mathematics (= r.m.).
6. Foundational aspects of r.m.
7. Uses of recursion theory in r.m.
8. An opportunity for the recursion theorists.
9. Reaction from the recursion theorists.
10. Symmetric ω-models of WKL_0.
11. Symmetric β-models.
12. Muchnik and Medvedev degrees of Π^0_1 subsets of 2^ω.

Foundations of mathematics (f.o.m.):

Foundations of mathematics is the study of the most basic concepts and logical structure of mathematics as a whole. Among the most basic mathematical concepts are:

number, set, function, algorithm, mathematical definition, mathematical proof, mathematical theorem, mathematical axiom.

Aristotle, Euclid, Descartes, Leibniz, . . . , Dedekind, Cantor, Frege, Russell, Zermelo, Hilbert, Weyl, Brouwer, Skolem, Gödel, Church, Turing, Post, Kleene, . . .

F.o.m. questions were the original motivation of both recursion theory (a.k.a. computability theory) and reverse mathematics.

Some key f.o.m. questions:

What is a computable function? What does it mean for a problem to be unsolvable? What are the appropriate axioms for mathematics?

Mathematical logic:

In the post-war period, f.o.m. evolved into a different subject, mathematical logic, which largely lost touch with its f.o.m. roots.

The 4 main subdivisions of mathematical logic are:

1. Model theory.

2. Set theory.

3. Recursion theory.

4. Proof theory.

For an overview, see *Handbook of Mathematical Logic*, edited by J. Barwise, 1977, XI + 1165 pages.

Each of the 4 has become isolated from the others. E.g., the panels on logic in the 20th and 21st centuries, at ASL 2000 in Urbana.

The FOM mailing list:

FOM is an automated e-mail list for discussing foundations of mathematics. There are currently more than 500 subscribers. There have been almost 5000 postings.

FOM is maintained and moderated by S. Simpson. The FOM Editorial Board consists of M. Davis, H. Friedman, C. Jockusch, D. Marker, S. Simpson, A. Urquhart.

FOM postings and information are available on the web at

`http://www.math.psu.edu/simpson/fom/`

Friedman and Simpson founded FOM in 1997, to promote a controversial idea: **mathematical logic is or ought to be driven by f.o.m. considerations.**

f.o.m. = foundations of mathematics.

History of recursion theory:

1930–1955:
Gödel, Turing, Church, Post, Kleene.
Motivated by f.o.m. considerations.
Unsolvability of the Halting Problem.
Unsolvability of the Entscheidungsproblem.
Turing degrees, i.e., degrees of unsolvability.
Recursive enumerability.

1944, Post's Problem:
Does there exist an r.e. degree of unsolvability different from that of the Halting Problem?

1956–1957:
Friedberg and Muchnik independently prove the existence of such degrees. They introduce a new and complicated method, *the priority method*.

For more than 50 years – until 1996 – the subject was known as "recursive function theory" or, for short, "recursion theory".

Early 1960's: Gerald Sacks.
The upper semi-lattice of r.e. Turing degrees.
Further structural results.
More complicated priority arguments.
Emphasis on methodology.
Finite injury versus infinite injury method.

Rejection of f.o.m. considerations.

"We regard an unsolved problem as interesting only if it seems likely that its solution requires a new trick." Sacks, *Degrees of Unsolvability*, 2nd edition, 1966, page 169.

"Remarks Against Foundational Activity", *Historia Mathematica*, 1975, pages 523–528.

Late 1960's and 1970's:
Sacks school (MIT and Harvard) pursue "generalized" or "higher" recursion theory.
Inspired by Kleene's hyperarithmetical theory, via Kreisel. Well-connected to the rest of mathematical logic, and to f.o.m.
"A crossroads of mathematical logic."

1970's: Robert Soare.
Rejects "generalized" recursion theory.
Pursues "classical" recursion theory.
A narrowing of the scope of r.t.
Renewed emphasis on methodology.
The $0'''$ priority method.

Recursion theory as an art or a sport.
Comparison: classical r.t. = Renaissance art,
generalized r.t. = Baroque art.
(Soare 1978, 2000)

1990's: Soare school (U of Chicago)
claim that priority methods are of great
importance in computer science.
Most computer scientists do not agree.

1996: Soare exercises leadership,
imposes wholesale change of terminology:
recursive becomes *computable*,
r.e. sets become *c.e. sets*, etc.
Bibliographic references are rewritten.
E.g., Cooper (MLQ, 2001, page 33)
changes the title of Post's 1944 paper.

History of reverse mathematics:

Kreisel 1960's introduces several subsystems of 2nd order arithmetic, including Δ^1_1-CA, Σ^1_1-AC, Σ^1_1-DC, Π^1_∞-TI_0.

Friedman 1967 (Ph.D. thesis, MIT) introduces a system equivalent to ATR, to show that Σ^1_1-AC $\neq$ Σ^1_1-DC.

Simpson 1973 (Berkeley) lectures on subsystems of 2nd order arithmetic and their role in f.o.m. Printed lecture notes.

Steel 1973 shows that ATR $\leftrightarrow$ comparability of countable well orderings, over Δ^1_1-CA. This and other r.m. results appear in Steel's Ph.D. thesis, supervised by Simpson.

Friedman 1974 (ICM lecture) states the first theme of reverse mathematics. Friedman 1975 (two JSL abstracts) introduces systems with restricted induction, including RCA_0.

History of r.m. (continued):

Simpson 1982 (Cornell recursion theory meeting) highlights the "big five": RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0.

Key r.m. papers by Simpson 1984 and Friedman/Simpson/Smith 1985 on reverse analysis, reverse algebra respectively.

Simpson from 1977 onward supervises numerous Ph.D. theses at Penn State: Smith, Brackin, Ferreira, Hirst, Brown, Yu, Marcone, Humphreys, Giusto, . . . , and publishes numerous papers . . . .

Simpson 1998 finishes his book on subsystems of 2nd order arithmetic and reverse mathematics.

Simpson 1999 begins assembling the companion volume, *Reverse Mathematics 2001*.

Two books on reverse mathematics:

1.

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Perspectives in Mathematical Logic
Springer-Verlag, 1999
XIV + 445 pages

`http://www.math.psu.edu/simpson/sosoa/`

2.

S. G. Simpson (editor)
Reverse Mathematics 2001

A volume of papers by various authors,
to appear in 2001,
approximately 400 pages.

`http://www.math.psu.edu/simpson/revmath/`

Current framework of reverse mathematics:

Second order arithmetic ($= \mathsf{Z}_2$) is a two-sorted system.

Number variables $m, n, \ldots$ range over

$$\omega = \{0, 1, 2, \ldots\} .$$

Set variables $X, Y, \ldots$ range over subsets of ω.

We have $+$, $\times$, $=$ on ω, plus the membership relation

$$\in \quad = \quad \{(n, X) : n \in X\} \quad \subseteq \quad \omega \times P(\omega) .$$

Within subsystems of Z_2, we can formalize contemporary rigorous mathematics (analysis, algebra, geometry, combinatorics, . . .).

Subsystems of Z_2 are the basis of our current understanding of the logical structure of contemporary mathematics.

Themes of reverse mathematics:

Let τ be a mathematical theorem. Let S_τ be the weakest natural subsystem of Z_2 in which τ is provable.

1. Very often, the principal axiom of S_τ is logically equivalent to τ.

2. Furthermore, only a few subsystems of Z_2 arise in this way.

For a full exposition, see my book *Subsystems of Second Order Arithmetic*, Springer, 1999, XIV + 445 pages.

Themes of r.m. (continued):

We develop a table indicating which mathematical theorems can be proved in which subsystems of Z_2.

	RCA_0	WKL_0	ACA_0	ATR_0	Π^1_1-CA_0
analysis (separable):					
differential equations	X	X			
continuous functions	X, X	X, X	X		
completeness, etc.	X	X	X		
Banach spaces	X	X, X			X
open and closed sets	X	X		X, X	X
Borel and analytic sets	X			X, X	X, X
algebra (countable):					
countable fields	X	X, X	X		
commutative rings	X	X	X		
vector spaces	X		X		
Abelian groups	X		X	X	X
miscellaneous:					
mathematical logic	X	X			
countable ordinals	X		X	X, X	
infinite matchings		X	X	X	
the Ramsey property			X	X	X
infinite games			X	X	X

The hierarchy of consistency strengths:

strong

- ⋮
- supercompact cardinal
- ⋮
- measurable cardinal
- ⋮
- ZFC (ZF set theory with choice)
- Zermelo set theory
- simple type theory

medium

- Z_2 (2nd order arithmetic)
- ⋮
- Π^1_2 comprehension
- Π^1_1 comprehension
- ATR_0 (arith. transfinite recursion)
- ACA_0 (arithmetical comprehension)

weak

- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- EFA (elementary arithmetic)
- bounded arithmetic
- ⋮

Foundational consequences of r.m.:

1. We precisely classify mathematical theorems, according to which subsystems of Z_2 they are provable in.

2. We identify certain subsystems of Z_2 as being mathematically natural. The naturalness is rigorously demonstrated.

3. We work out the consequences of particular foundational doctrines:

   - recursive analysis (Pour-El/Richards)
   - constructivism (Bishop)
   - finitistic reductionism (Hilbert)
   - predicativism (Weyl)
   - predicative reductionism (Feferman/Friedman/Simpson)
   - impredicative analysis (Takeuti/Schütte/Pohlers)

Foundational consequences (continued):

By means of reverse mathematics, we identify five particular subsystems of Z_2 as being mathematically natural. We correlate these systems to traditional f.o.m. programs.

RCA_0	constructivism	Bishop
WKL_0	finitistic reductionism	Hilbert
ACA_0	predicativism	Weyl, Feferman
ATR_0	predicative reductionism	Friedman, Simpson
Π^1_1-CA_0	impredicativity	Feferman et al.

We analyze these foundational proposals in terms of their consequences for mathematical practice. Under the various proposals, which mathematical theorems are "lost"?

Reverse mathematics provides precise answers to such questions.

Uses of recursion-theoretic methods in reverse mathematics:

Recursion-theoretic coding arguments have been used to obtain counterexamples in recursive algebra, recursive analysis, recursive combinatorics, etc.

See *Handbook of Recursive Mathematics*, two volumes, North-Holland, 1998, edited by Ershov/Goncharov/Nerode/Remmel.

Many of these coding arguments have been adapted and strengthened to obtain reversals a la reverse mathematics.

In all cases that I have looked at, priority arguments in this type of application are irrelevant or can easily be eliminated. Constructive content is thereby improved.

Uses of r.-t. methods, continued:

Recursion theory has been used to construct ω-models and β-models of subsystems of 2nd order arithmetic.

Turing ideals give ω-models of RCA_0.

Turing jump ideals give ω-models of ACA_0.

Basis theorems for Π^0_1 subsets of 2^ω give interesting ω-models of WKL_0. Recently, Medvedev and Muchnik degrees of nonempty Π^0_1 subsets of 2^ω have been extremely useful.

Basis theorems in hyperarithmetical theory give interesting β-models of ATR_0 and Π^1_∞-TI_0.

These techniques generalize to give wider classes of models, including non-ω-models, via an adaptation of generalized recursion theory. Compare “reverse recursion theory”.

Priority arguments are largely irrelevant.

Summary:

Reverse mathematics appears to be an excellent opportunity for recursion theory to reconnect with its f.o.m. roots. It is a rich problem area where recursion-theoretic methods can be applied to draw conclusions which are of general intellectual interest.

Reaction from recursion theorists:

A large number of prominent recursion theorists have reacted positively to r.m.

Cenzer, Cholak, Chong, Downey, Groszek, Harrington, Jockusch, Lempp, Remmel, Schmerl, Shore, Slaman, Steel, . . . have contributed to r.m. by proving substantial theorems and publishing substantial papers.

E.g., Richard Shore, while publicly expressing lack of enthusiasm for foundational or f.o.m. aspects, has also performed and supervised some excellent published research in r.m.

On the other hand, Robert Soare has reacted quite negatively to r.m. In an FOM posting of August 1999, Soare refers to:

> "[. . .] the HUGE GAPS in philosophy, approach, and value of mathematical items, between the REVERSE MATHEMATICIAN and the ACTIVE RESEARCHER in mathematics from TOPOLOGY to COMPUTABILITY THEORY. The former is eager to analyze the proof strength, axiomatic necessity, etc. of hypotheses but of mostly EXISTING theorems, while the working mathematician wants to get on with proving NEW theorems [. . .]"
>
> (Soare's emphasis)

implying that "reverse mathematicians" are not active researchers, and that their work is of relatively low value.

Reaction, continued:

In e-mail to his own department head (September 1999), Soare said:

> "I have spent a month contacting experts in logic and philosophy looking for weaknesses in reverse math. There are plenty. I am prepared to publicly take apart reverse math, its founders Friedman and Simpson, their research their publications and books, their fom [sic] list (for which I have a large number of strongly negative testimonials), and many other things about them. I have recruited helpers who will also send msgs [messages] infom [sic] and exfom [sic]. When this battle is over all in the communityh [sic] will be sick of hearing from any of us."

Why this extreme level of hostility?

Reaction, continued:

In private e-mail (September 1999), Soare informed me that he had spent a month gathering negative information about r.m. and had prepared a massive jihad against r.m. and me personally.

Why this extreme level of hostility?

This is more than the usual misunderstanding of the goals of f.o.m.

What happened here?

I think Soare was upset because of the success of FOM, interest shown in r.m., and interest shown in f.o.m. generally.

I think Soare finds r.m. particularly disturbing, because it uses many recursion-theoretic results and methods, but not those of his own specialty, r.e. sets and degrees and priority methods.

My current papers:

1. Simpson/Tanaka/Yamazaki (35 pages, 2000, to appear in APAL) contains many conservation results for WKL_0 over RCA_0.

2. Simpson (26 pages, 2000, to appear in Reverse Mathematics 2001) constructs models of WKL_0 where relative definability equals Turing reducibility.

3. Simpson (8 pages, 2000) constructs models of ATR_0 and of Π^1_∞-TI_0 where relative definability equals relative hyperarithmeticity.

4. Binns/Simpson (20 pages, 2001, under revision) obtains lattice embedding results for $\mathcal{P}_M$ and $\mathcal{P}_w$, the lattices of Medvedev and Muchnik degrees of Π^0_1 subsets of 2^ω.

5. Simpson (2001, in preparation) studies interesting subsets of $\mathcal{P}_M$ and $\mathcal{P}_w$.

An interesting ω-model of WKL_0:

Let $\mathcal{P}$ be the nonempty Π^0_1 subsets of 2^ω, ordered by inclusion. Forcing with $\mathcal{P}$ is known as Jockusch/Soare forcing.

Lemma (Simpson 2000). Let X be J/S generic. Suppose $Y \leq_T X$. Then (i) Y is J/S generic, and (ii) X is J/S generic relative to Y.

Theorem (Simpson 2000). There is an ω-model M of WKL_0 with the following property: For all $X, Y \in M$, X is definable from Y in M if and only if X is Turing reducible to Y.

Proof. M is obtained by iterated J/S forcing. We have

$$M = \mathsf{REC}[X_1, X_2, \ldots, X_n, \ldots]$$

where, for all n, X_{n+1} is J/S generic over $\mathsf{REC}[X_1, \ldots, X_n]$. To show that M has the desired property, we use symmetry arguments based on the Recursion Theorem.

Foundational significance of M:

The above ω-model, M, represents a compromise between the conflicting needs of

(a) recursive mathematics ("everything is computable")

and

(b) classical rigorous mathematics as developed in WKL_0 ("every continuous real-valued function on $[0,1]$ attains a maximum", "every countable commutative ring has a prime ideal", etc etc).

Namely, M contains enough nonrecursive objects for WKL_0 to hold, yet the recursive objects form the "definable core" of M.

Foundational significance (continued):

More generally, consider the scheme

> (*) For all X and Y, if X is definable from Y then X is recursive in Y.

in the language of 2nd order arithmetic.

Often in mathematics, under some assumptions on a given countably coded object X, there exists a unique countably coded object Y having some property stated in terms of X. In this situation, (*) implies that Y is Turing computable from X. This is of obvious f.o.m. significance.

Simpson 2000 shows that, for every countable model of WKL_0, there exists a countable model of $\mathsf{WKL}_0 + (*)$ with the same first order part. Thus $\mathsf{WKL}_0 + (*)$ is conservative over WKL_0 for first order arithmetical sentences.

Hyperarithmetical analogs:

Theorem (Simpson 2000). There is a countable β-model M such that, for all $X, Y \in M$, X is definable from Y in M if and only if X is hyperarithmetical in Y.

In the language of second order arithmetic, consider the scheme

> (**) for all X, Y, if X is definable from Y, then X is hyperarithmetical in Y.

Theorem (Simpson 2000).

1. $\mathsf{ATR}_0 + (**)$ is conservative over ATR_0 for Σ^1_2 sentences.
2. Π^1_∞-$\mathsf{TI}_0 + (**)$ is conservative over Π^1_∞-TI_0 for Σ^1_2 sentences.

Two new structures in recursion theory:

Recall that $\mathcal{P}$ is the set of nonempty Π^0_1 subsets of 2^ω.

$\mathcal{P}_w$ ($\mathcal{P}_M$) consists of the Muchnik (Medvedev) degrees of members of $\mathcal{P}$, ordered by Muchnik (Medvedev) reducibility.

P is Muchnik reducible to Q ($P \leq_w Q$) if for all $Y \in Q$ there exists $X \in P$ such that $X \leq_T Y$.

P is Medvedev reducible to Q ($P \leq_M Q$) if there exists a recursive functional $F : Q \to P$.

Note: $\leq_M$ is a uniform version of $\leq_w$.

$\mathcal{P}_w$ and $\mathcal{P}_M$ are countable distributive lattices with 0 and 1.

The lattice operations are given by

$$P \times Q = \{X \oplus Y : X \in P, Y \in Q\}$$

(least upper bound)

$$P + Q = \{\langle 0 \rangle^\frown X : X \in P\} \cup \{\langle 1 \rangle^\frown Y : Y \in Q\}$$

(greatest lower bound).

$P \equiv 0$ in $\mathcal{P}_w$ if and only if $P \cap \text{REC} \neq \emptyset$.

$P \equiv 0$ in $\mathcal{P}_M$ if and only if $P \cap \text{REC} \neq \emptyset$.

$P \equiv 1$ in $\mathcal{P}_w$, i.e., P is *Muchnik complete*, if and only if the Turing degrees of members of P are exactly the Turing degrees of complete extensions of PA. (Simpson 2001)

$P \equiv 1$ in $\mathcal{P}_M$, i.e., P is *Medvedev complete*, if and only if P is recursively homeomorphic to the set of complete extensions of PA. (Simpson 2000)

Trivially $P, Q > 0$ implies $P + Q > 0$, but we do not know whether $P, Q < 1$ implies $P \times Q < 1$.

In $\mathcal{P}_w$, for every $P > 0$, every countable distributive lattice is lattice embeddable below P. For $\mathcal{P}_M$ we have partial results in this direction.

To construct our lattice embeddings, we use infinitary "almost lattice" operations, defined in such a way that, if $\langle P_i : i \in \omega \rangle$ is a recursive sequence of members of $\mathcal{P}$, then

$$\prod_{i=0}^{\infty} P_i \qquad \text{and} \qquad \sum_{i=0}^{\infty} P_i$$

are again members of $\mathcal{P}$. We also use a finite injury priority argument a la Martin/Pour-El 1970 and Jockusch/Soare 1972. To push the embeddings below P, we use a Sacks preservation strategy.

This is ongoing joint work with my Ph. D. student Stephen Binns.

Corollary. In $\mathcal{P}_w$, for all $P >_w 0$ there exists Q such that $P >_w Q >_w 0$.
(nonexistence of minimal Muchnik degrees)

Corollary. In $\mathcal{P}_M$, for all $P >_M 0$ there exists Q such that $P >_M Q >_M 0$.
(nonexistence of minimal Medvedev degrees)

The last corollary was also obtained by Douglas Cenzer and Peter Hinman, using a different method: index sets.

Problem area:

Study structural properties of the countable distributive lattices $\mathcal{P}_w$ and $\mathcal{P}_M$. Lattice embeddings, extensions of embeddings, quotient lattices, cupping and capping, automorphisms, definability, decidability, etc.

An invidious comparison:

In some ways, the study of $\mathcal{P}_w$ and $\mathcal{P}_M$ parallels the study of $\mathcal{R}_T$, the Turing degrees of recursively enumerable subsets of ω.

Analogy: $$\frac{\mathcal{P}_w}{\mathcal{R}_T} = \frac{\mathsf{WKL}_0}{\mathsf{ACA}_0}$$

A regrettable aspect of $\mathcal{R}_T$ is that there are **no specific known examples** of recursively enumerable Turing degrees $\neq 0, 0'$. (See the extensive FOM discussion of July 1999, in the aftermath of the Boulder meeting.)

In this respect, $\mathcal{P}_w$ and $\mathcal{P}_M$ are **much better**.

Invidious comparison (continued):

For example, we have:

Theorem. The set of Muchnik degrees of Π^0_1 subsets of 2^ω of positive measure contains a maximum degree. This particular Muchnik degree is $\neq 0, 1$.

Question. What about Medvedev degrees?

The theorem follows from three known results.

1. $\{X : X$ is 1-random$\}$ is Σ^0_2 and of measure one. (Martin-Löf 1966)

2. $\{X : \exists Y \leq_T X\,(Y$ separates a recursively inseparable pair of r.e. sets$)\}$ is of measure zero. (Jockusch/Soare 1972)

3. If $P \in \mathcal{P}$ is of positive measure, then for all 1-random X there exists k such that $X^{(k)} = \lambda n.X(n+k) \in P$. (Kučera 1985)

A related but apparently new result:

Theorem (Simpson 2000). If X is 1-random and hyperimmune-free, then no $Y \leq_T X$ separates a recursively inseparable pair of r.e. sets.

Other related results:

1. If X is 1-random and of r.e. Turing degree, then X is Turing complete. (Kučera 1985)

2. $\{X : X$ is hyperimmune-free$\}$ is of measure zero. (Martin 1967, unpublished)

Foundational significance:

All of these results are informative with respect to ω-models of WWKL_0. WWKL_0 is a subsystem of second order arithmetic which arises in the Reverse Mathematics of measure theory. (Yu/Simpson 1990)

Some specific Medvedev degrees $\neq 0,1$:

For $k \geq 2$ let DNR_k be the set of k-valued DNR functions. Each DNR_k is recursively homeomorphic to a member of $\mathcal{P}$. DNR_2 is Medvedev complete. In $\mathcal{P}_M$ we have

$$\text{DNR}_2 >_M \text{DNR}_3 >_M \cdots >_M \sum_{k=2}^{\infty} \text{DNR}_k \,.$$

All of these Medvedev degrees are Muchnik complete. (Jockusch 1989)

Problem area:

Find additional natural examples of Medvedev and Muchnik degrees $\neq 0,1$.

Experience suggests that natural examples could be of significance for f.o.m.

A related problem of Reverse Mathematics:

Let DNR(k) be the statement that for all X there exists a k-valued DNR function relative to X. It is known that, for each $k \geq 2$, DNR(k) is equivalent to Weak König's Lemma over RCA_0. Is $\exists k\,(k \geq 2 \wedge \mathrm{DNR}(k))$ equivalent to Weak König's Lemma over RCA_0?

This has a bearing on graph coloring problems in Reverse Mathematics. See two recent papers of James H. Schmerl, to appear in MLQ and Reverse Mathematics 2001.

Another problem area:

One may study properties of interesting subsets of $\mathcal{P}_w$ and $\mathcal{P}_M$. For example, we may consider Muchnik and Medvedev degrees of $P \in \mathcal{P}$ with the following special properties:

1. P is of positive measure.

2. P is *thin*, i.e., for all Π^0_1 sets $Q \subseteq P$ there exists a clopen set $U \subseteq 2^\omega$ such that $P \cap U = Q$. (See also the recent paper of Cholak/Coles/Downey/Herrmann.)

3. P is *separating*, i.e.,

$$P = \{X \in 2^\omega : X \text{ separates } A, B\}$$

where A, B is a disjoint pair of r.e. sets.

These classes of Medvedev and Muchnik degrees are related in interesting ways.

Theorem (Simpson 2001). Let $P \subseteq 2^\omega$ be Π^0_1 of positive measure of maximum Muchnik degree. Let $Q \not\equiv_w 0$ be a thin Π^0_1 set. Then P and Q are Muchnik incomparable, i.e., $P \not\leq_w Q$ and $Q \not\leq_w P$.

Also, if P is as above and $Q \not\equiv_w 0$ is separating, then $Q \not\leq_w P$. (Jockusch/Soare 1972)

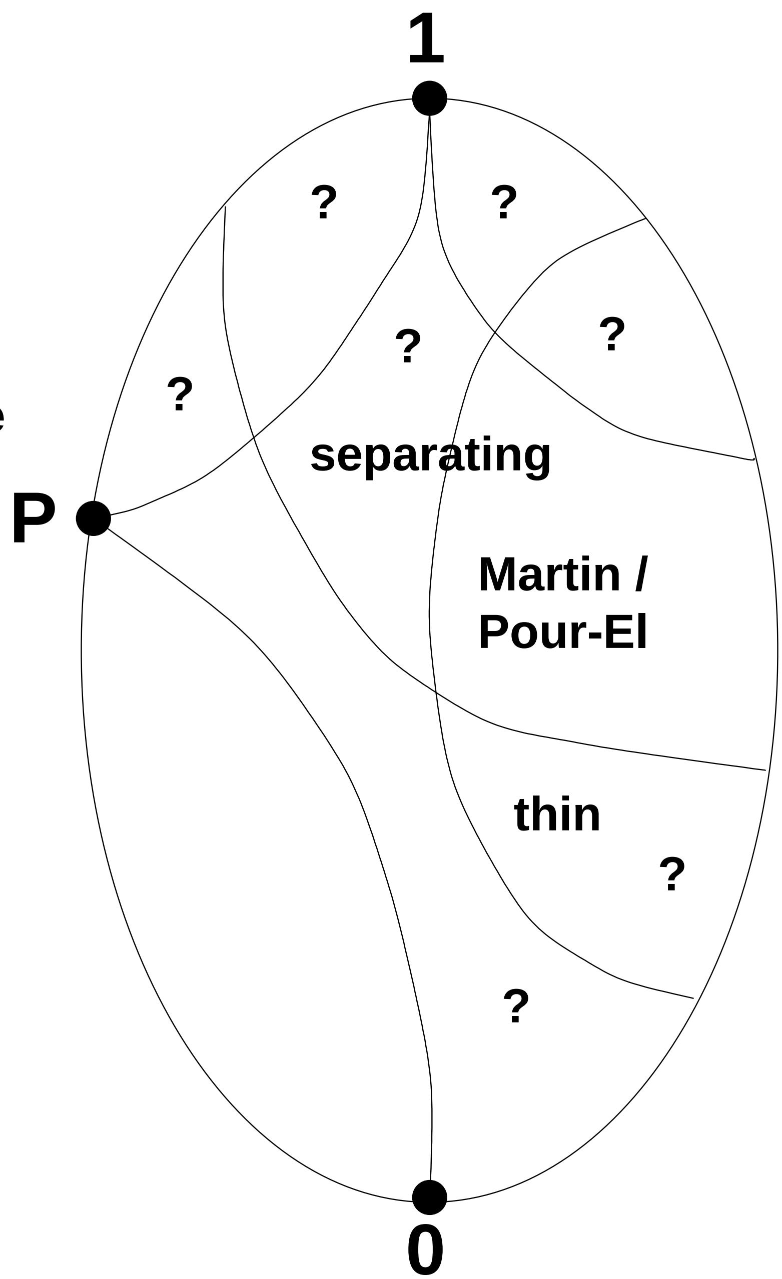
Theorem (Simpson 2001). Let P be as above. Then P is non-capping in $\mathcal{P}_w$. I.e., there do not exist $P_1, P_2 >_w P$ such that $P \equiv_w P_1 + P_2$, the infimum of P_1 and P_2.

A lemma used in proving these theorems:

Lemma. If $P, Q \in \mathcal{P}$ and $P \leq_w Q$, then there exists $R \subseteq Q$, $R \in \mathcal{P}$, such that $P \leq_M R$.

A picture of the Muchnik lattice $\mathcal{P}_w$:

maximum of
positive measure



Some of my papers are available at
`http://www.math.psu.edu/simpson/papers/`.

Transparencies for my talks are available at
`http://www.math.psu.edu/simpson/talks/`.

THE END