Reverse Mathematics:
An Overview

Stephen G. Simpson

Pennsylvania State University
http://www.math.psu.edu/simpson/
simpson@math.psu.edu

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Outline of this talk:

1. subsystems of Z_2 and SOSOA
2. f.o.m. and FOM
3. the Gödel hierarchy
4. r.m. as a classification program
5. r.m. and traditional foundational schemes (constructivism, predicativism, reductionist programs)
6. philosophical significance of r.m.
7. mathematical significance of r.m.
8. significance of r.m. for mathematical logic

History of reverse mathematics:

Kreisel in the 1960's introduced numerous subsystems of Z_2 including Δ^1_1-CA, Σ^1_1-AC, Σ^1_1-DC

Friedman 1967 introduced a system equivalent to ATR to show that Σ^1_1-AC $\neq$ Σ^1_1-DC.

Simpson 1973 lectured at Berkeley on subsystems of Z_2 and their role in f.o.m. Lecture notes are available.

Steel 1973 showed that ATR $\leftrightarrow$ comparability of countable well orderings, over Δ^1_1-CA. This and other r.m. results appeared in Steel's Ph.D. thesis, supervised by me.

Friedman in his 1974 ICM lecture stated the theme of reverse mathematics. In JSL abstracts he introduced systems with restricted induction.

History of r.m. (continued):

Simpson in a lecture at a 1982 recursion theory meeting (Cornell) isolated the "big five": RCA_0, WKL_0, ACA_0, ATR_0, Π^1_1-CA_0.

Papers by Simpson and Friedman/Simpson/Smith.

Simpson supervises Ph.D. theses at Penn State: Smith, Brackin, Ferreira, Hirst, Brown, Yu, Marcone, Humphreys, Giusto, . . . ,
+ numerous papers . . . . (1979–1998)

Simpson 1998 finally finishes his book on subsystems of Z_2 and reverse mathematics.
(Sorry for the delay!)

Simpson 1999 begins assembling the volume *Reverse Mathematics 2001*.

Book on reverse mathematics:

Stephen G. Simpson
Subsystems of Second Order Arithmetic
Perspectives in Mathematical Logic
Springer-Verlag, 1999
XIV + 445 pages

`http://www.math.psu.edu/simpson/sosoa/`

Order: 1-800-SPRINGER

List price: $60

Discount: 30 percent for ASL members, mention promotion code S206

Unfortunately the book is no longer available from Springer, but there is hope that the ASL will reprint it soon.

Another book on reverse mathematics:

S. G. Simpson (editor)
Reverse Mathematics 2001

A volume of papers by various authors,
to appear in 2001,
approximately 400 pages.

`http://www.math.psu.edu/simpson/revmath/`

An upcoming reverse mathematics event:

ASL/APA Meeting, Minneapolis, May 3–5, 2001. Panel Discussion on "Computability Theory and Reverse Mathematics", organized by M. Detlefsen. Panelists: J. Hirst, S. Simpson, R. Solomon.

People here in Philadelphia who have contributed to reverse mathematics:

Avigad, Cenzer, Feng, Friedman, Girard, Hirschfeldt, Hirst, Jockusch, Lempp, Schmerl, Simpson, Shore, Slaman, Solomon

Foundations of mathematics (f.o.m.):

Foundations of mathematics is the study of the most basic concepts and logical structure of mathematics as a whole. Among the basic concepts are: number, set, function, algorithm, mathematical proof, mathematical definition, mathematical axiom.

A key f.o.m. question:

What are the appropriate axioms for mathematics?

The FOM mailing list:

FOM is an automated e-mail list for discussing foundations of mathematics. There are currently more than 500 subscribers. There have been more than 4800 postings.

FOM is maintained and moderated by S. Simpson. The FOM Editorial Board consists of M. Davis, H. Friedman, C. Jockusch, D. Marker, S. Simpson, A. Urquhart.

FOM postings and information are available on the web at

`http://www.math.psu.edu/simpson/fom/`

Friedman and Simpson founded FOM in order to promote a controversial idea: **mathematical logic is or ought to be driven by f.o.m. considerations.**

f.o.m. = foundations of mathematics.

Background of reverse mathematics:

Second order arithmetic (Z_2) is a two-sorted system.

Number variables $m, n, \ldots$ range over

$$\omega = \{0, 1, 2, \ldots\} \ .$$

Set variables $X, Y, \ldots$ range over subsets of ω.

We have $+$, $\times$, $=$ on ω, plus the membership relation

$$\in \quad = \quad \{(n, X) : n \in X\} \quad \subseteq \quad \omega \times P(\omega) \ .$$

Within subsystems of second order arithmetic, we can formalize rigorous mathematics (analysis, algebra, geometry, . . .).

Subsystems of second order arithmetic are the basis of our current understanding of the logical structure of contemporary mathematics.

Themes of reverse mathematics:

Let τ be a mathematical theorem. Let S_τ be the weakest natural subsystem of second order arithmetic in which τ is provable.

1. Very often, the principal axiom of S_τ is logically equivalent to τ.

2. Furthermore, only a few subsystems of second order arithmetic arise in this way.

For a full exposition, see my book.

Themes of r.m. (continued):

We develop a table indicating which mathematical theorems can be proved in which subsystems of Z_2.

	RCA_0	WKL_0	ACA_0	ATR_0	Π^1_1-CA_0
analysis (separable):					
differential equations	X	X			
continuous functions	X, X	X, X	X		
completeness, *etc.*	X	X	X		
Banach spaces	X	X, X			X
open and closed sets	X	X		X, X	X
Borel and analytic sets	X			X, X	X, X
algebra (countable):					
countable fields	X	X, X	X		
commutative rings	X	X	X		
vector spaces	X		X		
Abelian groups	X		X	X	X
miscellaneous:					
mathematical logic	X	X			
countable ordinals	X		X	X, X	
infinite matchings		X	X	X	
the Ramsey property			X	X	X
infinite games			X	X	X

The hierarchy of consistency strengths:

strong:

- ⋮
- supercompact cardinal
- ⋮
- measurable cardinal
- ⋮
- ZFC (ZF set theory with choice)
- Zermelo set theory
- simple type theory

medium:

- Z_2 (2nd order arithmetic)
- ⋮
- Π^1_2 comprehension
- Π^1_1 comprehension
- ATR_0 (arith. transfinite recursion)
- ACA_0 (arithmetical comprehension)

weak:

- WKL_0 (weak König's lemma)
- RCA_0 (recursive comprehension)
- PRA (primitive recursive arithmetic)
- EFA (elementary arithmetic)
- bounded arithmetic
- ⋮

Foundational consequences of r.m.:

1. We precisely classify mathematical theorems, according to which subsystems of Z_2 they are provable in.

2. We identify certain subsystems of Z_2 as being mathematically natural.
   The naturalness is rigorously demonstrated.

3. We work out the consequences of particular foundational doctrines:

   - recursive analysis (Pour-El/Richards)
   - constructivism (Bishop)
   - finitistic reductionism (Hilbert)
   - predicativism (Weyl)
   - predicative reductionism (Feferman/Friedman/Simpson)
   - impredicative analysis (Takeuti/Schütte/Pohlers)

Foundational consequences (continued):

By means of reverse mathematics, we identify five particular subsystems of Z_2 as being mathematically natural. We correlate these systems to traditional f.o.m. programs.

RCA_0	constructivism	Bishop
WKL_0	finitistic reductionism	Hilbert
ACA_0	predicativism	Weyl, Feferman
ATR_0	predicative reductionism	Friedman, Simpson
Π^1_1-CA_0	impredicativity	Feferman *et al.*

We analyze these foundational proposals in terms of their consequences for mathematical practice. Specifically, under the various proposals, which mathematical theorems are "lost"? Reverse mathematics provides precise answers to such questions.

Significance of r.m. for mathematical logic:

1. **recursion theory**
   - reversals use coding arguments
   - ω-models of subsystems of Z_2
   - Turing ideals (RCA_0)
   - Turing jump ideals (ACA_0)
   - Π^0_1 subsets of 2^ω (WKL_0)
   - hyperarithmetic theory (ATR_0, Π^1_1-CA_0)
   - basis and anti-basis theorems
   - Medvedev and Muchnik degrees

2. **model theory**
   - models of subsystems of Z_2
   - ω-models and non-ω-models
   - nonstandard models of arithmetic
   - model-theoretic conservation results

3. **set theory**
   - r.m. highlights the foundational significance of set theory
   - ZFC is usually too strong for math, but r.m. highlights the rare cases when strong axioms are actually needed
   - set-theoretic methods are used to build models of subsystems of Z_2 (constructible sets, forcing, etc)

4. **proof theory**
   - subsystems of Z_2 studied in proof theory
   - r.m. brings out their f.o.m. significance
   - Gentzen-style ordinal analysis
   - proof-theoretic conservation results

My recent papers:

1. Simpson/Tanaka/Yamazaki (35 pages, 2000, submitted) contains many conservation results for WKL_0 over RCA_0.

2. Simpson (26 pages, 2000, to appear in Reverse Mathematics 2001) constructs models of WKL_0 where relative definability equals Turing reducibility.

3. Simpson (8 pages, 2000) constructs β-models where relative definability equals relative hyperarithmeticity.

4. Binns/Simpson (20 pages, 2001, under revision) obtain lattice embedding results for $\mathcal{P}_M$ and $\mathcal{P}_w$, the lattices of Medvedev and Muchnik degrees of nonempty Π^0_1 subsets of 2^ω.

5. Simpson (2001, in preparation) studies interesting subsets of $\mathcal{P}_M$ and $\mathcal{P}_w$.