Combining basis theorems

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Basis theorems.

A basis theorem is a theorem of the form:

> For any nonempty effectively closed set in Euclidean space, at least one member of the set is "close to being computable".

Some well known basis theorems are:

- the Low Basis Theorem,
- the R.E. Basis Theorem,
- the Hyperimmune-Free Basis Theorem,
- the Cone Avoidance Basis Theorem,
- the Randomness Preservation Basis Thm.

Less well known is a basis theorem of Higuchi/Hudelson/Simpson/Yokoyama on preservation of partial randomness.

Basis theorems are important for applications to the foundations of mathematics: models of arithmetic, Scott sets, ω-models of WKL_0, etc.

We discuss the possibilities for combining these basis theorems.

Three basis theorems.

Let $\leq_T$ denote Turing reducibility.

Let $'$ denote the Turing jump operator.

The Low Basis Theorem:

For any nonempty effectively closed set Q, there exists $Z \in Q$ such that $Z' \leq_T 0'$.

The R.E. Basis Theorem:

For any nonempty effectively closed set Q, there exists $Z \in Q$ such that Z is of recursively enumerable Turing degree.

We say that Z is *hyperimmune-free* if ($\forall$ functions $f \leq_T Z$) ($\exists$ recursive function g) $\forall n\,(f(n) < g(n))$.

The Hyperimmune-Free Basis Theorem:

For any nonempty effectively closed set Q, $(\exists Z \in Q)$ (Z is hyperimmune-free).

These three basis theorems are due to Jockusch/Soare 1972.

Can we combine these basis theorems?

No. The Jockusch/Soare basis theorems are known to be "pairwise incompatible."

1. The Arslanov Completeness Criterion provides a nonempty effectively closed Q such that for all r.e. sets A, if $(\exists Z \in Q)\,(Z \leq_{\mathrm{T}} A)$ then $0' \leq_{\mathrm{T}} A$.

   Therefore, the Low Basis Theorem and the R.E. Basis Theorem cannot be combined into one basis theorem.

2. It is known that for hyperimmune-free Z one cannot have $0 <_{\mathrm{T}} Z \leq_{\mathrm{T}} 0'$.

   Therefore, the Hyperimmune-Free Basis Theorem cannot be combined with the Low Basis Theorem or with the R.E. Basis Theorem.

Two more basis theorems.

The Cone Avoidance Basis Theorem:

For any nonempty effectively closed set Q, if $A \not\leq_{\mathrm{T}} 0$ then $(\exists Z \in Q)\,(A \not\leq_{\mathrm{T}} Z)$.

More generally, if $\forall i\,(A_i \not\leq_{\mathrm{T}} 0)$ then $(\exists Z \in Q)\,\forall i\,(A_i \not\leq_{\mathrm{T}} Z)$.

Gandy/Kreisel/Tait, 1960.

Let $\mathsf{MLR} = \{X \mid X$ is Martin-Löf random$\}$. Let $\mathsf{MLR}^Z = \{X \mid X$ is Martin-Löf random relative to $Z\}$.

The Randomness Preservation Basis Theorem:

For any nonempty effectively closed set Q, if $X \in \mathsf{MLR}$ then $(\exists Z \in Q)\,(X \in \mathsf{MLR}^Z)$.

Reimann/Slaman, not yet published.

Downey/Hirschfeldt/Miller/Nies, 2005.

Simpson/Yokoyama, 2011.

More combinations of basis theorems?

It is known that Cone Avoidance can be combined with the Low Basis Theorem, or with the Hyperimmune-free Basis Theorem, but not with the R.E. Basis Theorem. (See for instance Downey/Hirschfeldt §2.19.3.)

Also, Randomness Preservation cannot be combined with the Low or the R.E. or the Hyperimmune-Free Basis Theorem.

Specifically, let $\Omega \in \mathsf{MLR}$ be such that $\Omega \equiv_{\mathsf{T}} 0'$. It is known that such reals exist (Chaitin, Kučera/Gács). We then have:

1. Any $Z \leq_{\mathsf{T}} 0'$ such that $\Omega \in \mathsf{MLR}^Z$ is K-trivial, hence not PA-complete. (See Chapter 11 of Downey/Hirschfeldt 2010 or Chapter 5 of Nies 2009.)

2. Any hyperimmune-free Z such that $\Omega \in \mathsf{MLR}^Z$ is recursive. (See Theorem 8.1.18 of Nies 2009.)

Combining basis theorems.

	Low	R.E.	H.I.F.	C.A.	R.P.
Low	1	0	0	1	0
R.E.	0	1	0	0	0
H.I.Free	0	0	1	1	0
Cone Av.	1	0	1	1	**???**
Rand. Pres.	0	0	0	**???**	1

Remaining question: Can Cone Avoidance be combined with Randomness Preservation?

The answer to this question involves LR-reducibility.

Define $A \leq_{\mathrm{LR}} B \iff \mathrm{MLR}^B \subseteq \mathrm{MLR}^A$. Clearly $A \leq_{\mathrm{T}} B$ implies $A \leq_{\mathrm{LR}} B$, and it is known that $A \leq_{\mathrm{LR}} 0$ implies $A' \leq_{\mathrm{T}} 0'$. A major theorem of Nies is that $A \leq_{\mathrm{LR}} 0 \iff A$ is K-trivial. See Nies 2009 or Downey/Hirschfeldt 2010.

A theorem which combines Cone Avoidance and Randomness Preservation:

Theorem 1 (Simpson/Stephan, 2013)**.**
For any nonempty effectively closed set Q, if $X \in \mathsf{MLR}$ and $\forall i\,(A_i \not\leq_{\mathsf{LR}} 0$ or $A_i \not\leq_{\mathsf{T}} X)$, then $(\exists Z \in Q)\,(X \in \mathsf{MLR}^Z$ and $\forall i\,(A_i \not\leq_{\mathsf{T}} Z))$.

On the other hand, let $\Omega \in \mathsf{MLR}$ be such that $\Omega \equiv_{\mathsf{T}} 0'$. It is well known that such reals exist (Chaitin, Kučera/Gács).

Theorem 2 (Simpson/Stephan, 2013)**.**
$\exists$ nonempty effectively closed set Q such that $(\forall A \leq_{\mathsf{LR}} 0)\,(\forall Z \in Q)\,(\Omega \in \mathsf{MLR}^Z \Rightarrow A \leq_{\mathsf{T}} Z)$.

The proof uses a result of Miller 2010.

Summary of Theorems 1 and 2:
Randomness Preservation cannot be combined with Cone Avoidance, but only because $A \not\leq_{\mathsf{T}} 0$ does not imply $A \not\leq_{\mathsf{LR}} 0$.

Preservation of partial randomness.

Let $f : \{0,1\}^* \to [-\infty,\infty]$ be an arbitrary recursive function.

For $S \subseteq \{0,1\}^*$ let $\text{wt}_f(S) = \sum_{\sigma \in S} 2^{-f(\sigma)}$, $\text{pwt}_f(S) = \sup\{\text{wt}_f(P) \mid P \subseteq S \text{ prefix-free}\}$, and $[\![S]\!] = \{X \in \{0,1\}^{\mathbb{N}} \mid (\exists \sigma \in S)\,(\sigma \subset X)\}$.
We say that X is <u>strongly f-random</u> if $X \notin \bigcap_n [\![S_n]\!]$ for all uniformly r.e. $S_n \subseteq \{0,1\}^*$ such that $\forall n\,(\text{pwt}_f(S_n) \leq 2^{-n})$.

Martin-Löf randomness is the special case $f(\sigma) = |\sigma|$. In this case $\text{pwt}_f(S) = \mu([\![S]\!])$ where μ is the fair coin measure on $\{0,1\}^{\mathbb{N}}$.

<u>Partial Randomness Preservation</u>:
For any nonempty effectively closed set Q, if X is strongly f-random then $(\exists Z \in Q)$ (X is strongly f-random relative to Z).

More generally, if $\forall i\,(X_i$ is strongly f_i-random) then $(\exists Z \in Q)\,\forall i\,(X_i$ is strongly f_i-random relative to Z).

Higuchi/Hudelson/Simpson/Yokoyama, 2012.

To what extent can we combine
Partial Randomness Preservation
with Cone Avoidance?

Theorem 3 (implicit in H/H/S/Y 2012). For any nonempty effectively closed set Q, if $\forall i\,(A_i \not\leq_{\mathsf{LR}} 0$ and X_i is strongly f_i-random), then $(\exists Z \in Q)\,\forall i\,(A_i \not\leq_{\mathsf{LR}} Z$ and X_i is strongly f_i-random relative to Z).

On the other hand, because of Theorem 2, we cannot always replace $\leq_{\mathsf{LR}}$ by $\leq_{\mathsf{T}}$.

Can we sometimes replace $\leq_{\mathsf{LR}}$ by $\leq_{\mathsf{T}}$?

A typical open question:

Define X to be strongly half-random $\iff$ X is strongly f-random where $f(\sigma) = |\sigma|/2$.

If Q is nonempty effectively closed, and if $A \not\leq_{\mathsf{T}} 0$ and X is strongly half-random, does there exist $Z \in Q$ such that $A \not\leq_{\mathsf{T}} Z$ and X is strongly half-random relative to Z?

Proofs of Theorems 1 and 2.

To prove Theorem 1, we use
the Cone Avoidance Basis Theorem,
relativized to X.

To prove Theorem 2, we use
K = prefix-free Kolmogorov complexity.

(1) If $\Omega \in \mathsf{MLR}^Z$ then $|\mathsf{K}(n) - \mathsf{K}^Z(n)| \leq O(1)$ for infinitely many n. (Miller 2010.)

(2) If $\Omega \in \mathsf{MLR}^Z$ and Z is PA-complete, then there exist a Z-recursive function F and an infinite Z-recursive set A such that $|\mathsf{K}(n) - F(n)| \leq O(1)$ for all $n \in A$.

(3) Let C = plain Kolmogorov complexity. Chaitin 1976 proved: every C-trivial real is computable. Using F and A as in (2), we similarly prove: every K-trivial real is $\leq_{\mathsf{T}} Z$.

Theorems 1 and 2 are in Simpson/Stephan, 2013, in preparation.

Recent literature.

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Thank you for your attention!