Schnorr Randomness and the Lebesgue Differentiation Theorem

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Motivation

The Lebesgue Differentiation Theorem states:

Theorem

Given $f \in L_1([0,1]^d)$ we can find a null set S such that

$$f(x) \;=\; \lim_{Q \searrow x} \frac{1}{\mu(Q)} \int_Q f \, d\mu \tag{1}$$

for all $x \notin S$. The limit is taken over all cubes Q containing x as the diameter of Q tends to 0. Here μ is Lebesgue measure.

Question: What can we say about this null set?

Effectively Open Sets

Definition

1. A set $U \subseteq [0,1]^d$ is Σ_1^0 if it is effectively open, i.e

$$U = \bigcup_{i=0}^{\infty} B(a_i, r_i)$$

where $B(a_i, r_i)$, $i = 0, 1, \ldots$ is a computable sequence of rational balls.

2. A sequence of sets $U_n \subseteq [0,1]^d$, $n = 0, 1, 2, \ldots$ is *uniformly* Σ_1^0 if

$$U_n = \bigcup_{i=0}^{\infty} B(a_{ni}, r_{ni})$$

for all n, where $B(a_{ni}, r_{ni})$, $n = 0, 1, 2, \ldots$, $i = 0, 1, 2, \ldots$ is a computable double sequence of rational balls.

Randomness

Definition

A *Martin-Löf test* is a uniformly Σ^0_1 sequence of sets U_n, $n = 0, 1, 2, \ldots$ such that $\mu(U_n) \leq 1/2^n$ for all $n \in \mathbb{N}$. A point $x \in [0,1]^d$ is said to *pass the test* if $x \notin \bigcap_{n=0}^{\infty} U_n$. We say that x is *Martin-Löf random* if it passes every Martin-Löf test.

Definition

A *Schnorr test* is a Martin-Löf test, U_n, $n = 0, 1, 2, \ldots$, such that $\mu(U_n)$ is uniformly computable for all n. We say that x is *Schnorr random* if it passes every Schnorr test.

L_1-Computability

Definition

A function $f \in L_1([0,1]^d)$ is L_1*-computable* if there exists a computable sequence of polynomials with rational coefficients, denoted f_n, such that

$$\|f - f_n\|_1 \leq \frac{1}{2^n}$$

for all n.

Theorem (Rough Statement)

$x \in [0,1]^d$ is Schnorr random if and only if for all L_1-computable $f : [0,1]^d \to \mathbb{R}$,

$$f(x) = \lim_{Q \searrow x} \frac{1}{\mu(Q)} \int_Q f \, d\mu.$$

Representative of f

Definition

Let $f : [0,1]^d \to \mathbb{R}$ be L_1-computable. Then by the previous lemma, we can define a new function $\widehat{f} : [0,1]^d \to \mathbb{R}$,

$$\widehat{f}(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

Remark

Note that $\widehat{f}$ is L_1-equivalent to f.

L_1-computable functions $\to$ Schnorr tests

Theorem (P/Rojas/Simpson 2012, [3])

Let $f : [0,1]^d \to \mathbb{R}$ be an L_1-computable function.

1. There exists a Schnorr test such that for all x that pass the test, $\lim_{n\to\infty} f_n(x)$ exists.
2. There exists a Schnorr test such that for all x that pass the test,
$$\lim_{n\to\infty} f_n(x) = \widehat{f}(x) = \lim_{Q\searrow x} \frac{1}{\mu(Q)} \int_Q f \, d\mu.$$

Schnorr tests $\to$ L_1-computable functions

Theorem (P/Rojas/Simpson 2012, [3])

Let U_n, $n = 0, 1, 2, \ldots$ be a Schnorr test. Then, there exists an L_1-computable function $f : [0,1]^d \to \mathbb{R}$ such that for all $x \in \bigcap_{n\in\mathbb{N}} U_n$,

$$\lim_{Q \searrow x} \frac{1}{\mu(Q)} \int_Q f \, d\mu$$

does not exist.

Related Work

Theorem (Brattka/Miller/Nies 2011, [1])

Let $z \in [0,1]$. Then,

1. z is computably random if and only if every nondecreasing computable function $f : [0,1] \to \mathbb{R}$ is differentiable at z.
2. z is Martin-Löf random if and only if every computable function $f : [0,1] \to \mathbb{R}$ of bounded variation is differentiable at z.
3. z is weakly 2-random if and only if every almost everywhere differentiable computable function $f : [0,1] \to \mathbb{R}$ is differentiable at z.

Related Work-continued

Theorem (Freer/Kjos-Hanssen/Nies/Stephan 2013, [2])

Let $z \in [0,1]$. Then z is computably random if and only if each computable Lipschitz function $f : [0,1] \to \mathbb{R}$ is differentiable at z.

Theorem (Freer/Kjos-Hanssen/Nies/Stephan 2013, [2])

Let $z \in [0,1]^n$ and $p \geq 1$ be a computable real. Then z is Schnorr random if and only if z is a Lebesgue point of each L_p-computable bounded function $g : [0,1]^n \to \mathbb{R}$.

Bibliography I

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